Online Appendix to "Negative Nominal Interest Rates and Monetary Policy"

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In this online appendix, I label equation (i) in the body of the paper as equation (Pi). For example, (P5) is equation (5) in the paper.

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A First-Order Conditions for Private Bank's Problem

Let λ denote the Lagrange multiplier associated with the collateral constraint (P11). Then, the first-order conditions for the bank's problem (P9), subject to (P10)-(P12), are given by

$$\frac{\beta[1-\alpha^s+\alpha^s(1-\alpha^b)\eta]}{\pi}u'\left([1-\alpha^s+\alpha^s(1-\alpha^b)\eta]\frac{\beta c'}{\pi}-\mu\right)-\eta-\frac{\lambda\delta}{\pi}=0, \tag{1}$$

$$\beta u'(\beta d)-\beta-\lambda=0, \tag{2}$$

$$-1+\frac{\beta R^m}{\pi}+\frac{\lambda R^m(1-\delta)}{\pi}=0, \tag{3}$$

$$-1+\frac{\beta R^b}{\pi}+\frac{\lambda R^b(1-\delta)}{\pi}=0, \tag{4}$$

$$-\eta+\frac{\beta}{\pi}-\beta\gamma+\frac{\lambda(1-\delta)}{\pi}\leq 0, \tag{5}$$

$$\lambda\left[-(1-\rho)d+\frac{(1-\delta)\left(R^m m+R^b b+c\right)}{\pi}-\frac{\rho c'}{\pi}\right]=0. \tag{6}$$

B Equilibrium with Deflation and No Theft

In this section, I focus on stationary equilibria characterized by deflation and a sufficiently high cost of theft ε. With deflation, the real value of currency increases over time, allowing private banks to acquire a sufficient amount of currency in the CM. As private banks do not need to withdraw currency from the central bank, the price of currency is not determined by the non-par exchange rate η. In this case, sellers become indifferent between depositing their currency with the central bank and engaging in side-trading with private banks. Thus, in equilibrium, one unit of real currency is exchanged for one unit of goods in the CM. From (P7) and Lemma 1, a necessary condition for this equilibrium to exist is given by

$$\varepsilon\geq\frac{\rho(x^c+\mu)}{\beta}.$$

As the price of real currency is one, the values of η in equilibrium conditions (P13)-(P17) and (1)-(6) must be replaced by one. Then, from (P25), the effective lower bound (ELB) on nominal interest rates R^m can be defined as

$$R^m \geq \frac{1}{1+\beta\gamma} \equiv ELB.$$

This implies that the non-par exchange rate η is irrelevant to the ELB. The non-par exchange rate policy is ineffective because, if it is introduced, private banks would stop withdrawing currency from the central bank. Also, it is obvious that the non-par exchange rate would not affect equilibrium prices and allocations.

The inflation rate in this equilibrium can be written as

$$\pi = \beta \left[u'(x^c) - \delta u'(x^d) + \delta \right].$$

To observe deflation in equilibrium, both β and $u'(x^c)$ must be sufficiently low, while δ and $u'(x^d)$ must be sufficiently high. It turns out that v must be sufficiently high and R^m must be sufficiently low to support deflation in equilibrium. Note that, in the main body of the paper, I focus on cases with sufficiently low v and γ to ensure that there is inflation in equilibrium for any R^m higher than the ELB.

C Equilibrium with Potential Disintermediation

In this section, I formally characterize the extended model introduced in Section 5 on Disintermediation. In Sections 2-4, I assumed that a fraction ρ of buyers must use currency, while the remaining buyers must use non-currency assets. In contrast, the extended model allows buyers to opt out of banking arrangements and use currency in all DM transactions. This implies that the fraction $1-\rho$ of buyers can choose between currency and bank deposits, while the fraction ρ continues to use currency as in the baseline model. Apart from this modification, the model remains the same as described in Section 2.

C.1 Optimization

In this setting, some buyers may find it optimal to opt out of banking arrangements. Let θ denote the fraction of buyers who choose to deposit with private banks in the CM. Each bank's contracting problem and first-order conditions remain identical to those in the baseline model. Meanwhile, the fraction $1-\theta$ of buyers opt out of banking arrangements and choose the optimal quantity of currency c^o to solve the following problem:

$$\max_{c^o \geq 0} \left\{ -\eta c^o + u\left(\frac{\beta c^o \left[1-\alpha^s+\alpha^s(1-\alpha^b)\eta\right]}{\pi} - \mu \right) \right\}. \tag{7}$$

Although banks do not write deposit contracts with these buyers, they withdraw currency from the central bank to sell it to these buyers whenever necessary. With inflation, the price of currency in equilibrium is η, as in the baseline model. The first-order condition of the above problem is given by

$$-\eta + \frac{\beta\left[1-\alpha^s+\alpha^s(1-\alpha^b)\eta\right]}{\pi} u'\left(\frac{\beta c^o \left[1-\alpha^s+\alpha^s(1-\alpha^b)\eta\right]}{\pi} - \mu \right) = 0. \tag{8}$$

Let x^o denote the consumption quantity in the DM for these non-contracting buyers. The asset market clearing conditions are

$$\theta c + (1-\theta)c^o = \bar{c}; \qquad \theta m = \bar{m}; \qquad \theta b = \bar{b}. \tag{9}$$

Let U^b and U^o denote the expected utilities for buyers who obtain banking contracts and those who opt out, respectively. Using the first-order conditions for the bank's problem and equations (P10) and (8), I obtain

$$U^b = \rho\left[u(x^c) - (x^c+\beta\mu)u'(x^c)\right] + (1-\rho)\left[u(x^d) - x^d u'(x^d)\right], \tag{10}$$

$$U^o = u(x^o) - (x^o+\beta\mu)u'(x^o). \tag{11}$$

In equilibrium, the fraction θ must solve the following problem:

$$\max_{0\le\theta\le1}\left[\theta U^b + (1-\theta)U^o\right], \tag{12}$$

which implies that buyers' participation in banking arrangements, represented by θ, must be consistent with their utility maximization problem.

Depending on whom each seller met in the previous DM, the quantity of currency holdings in the TS varies across sellers. Buyers who hold deposit contracts withdraw c' units of currency (in real terms) to purchase goods in the DM. Thus, sellers who meet these buyers in the DM will hold $\frac{c'}{\pi}$ units of currency in the following TS. Meanwhile, buyers who do not hold deposit contracts have c^o units of currency. Therefore, sellers who meet these buyers in the DM will hold $\frac{c^o}{\pi}$ units of currency in the following TS. For simplicity, I assume that each seller decides whether to carry currency into the TS before knowing the type of the buyer they will meet in the DM.

The fraction of buyers acquiring theft technology, α^b, and the fraction of sellers carrying currency into the TS, α^s, must be incentive-compatible in equilibrium:

$$\alpha^b = \begin{cases} 0 & \text{if } \varepsilon > \frac{[\theta\rho c' + (1-\theta)c^o]\alpha^s\eta}{\pi} \\ \in [0,1] & \text{if } \varepsilon = \frac{[\theta\rho c' + (1-\theta)c^o]\alpha^s\eta}{\pi} \\ 1 & \text{if } \varepsilon < \frac{[\theta\rho c' + (1-\theta)c^o]\alpha^s\eta}{\pi} \end{cases} \tag{13}$$

With probability $\theta\rho\alpha^s$, a buyer is matched with a seller holding $\frac{c'}{\pi}$ units of currency, and with probability $(1-\theta)\alpha^s$, the buyer meets a seller holding $\frac{c^o}{\pi}$ units of currency. The above condition states that theft does not occur if it is too costly, occurs sometimes if the buyer is indifferent between stealing and not stealing, and always occurs if it is strictly profitable.

Also, each seller's decision on whether to carry currency into the TS must be incentive-compatible:

$$\alpha^s = \begin{cases} 0 & \text{if } (1-\alpha^b)\eta < 1 \\ \in [0,1] & \text{if } (1-\alpha^b)\eta = 1 \\ 1 & \text{if } (1-\alpha^b)\eta > 1 \end{cases} \tag{14}$$

C.2 Effective Lower Bound on Nominal Interest Rates

I will focus on cases where the cost of theft ε is sufficiently small to sustain a theft equilibrium.[1] Additionally, I will consider scenarios where the supply of collateralizable assets is insufficient to support the first-best level of consumption in deposit-based DM transactions, as in previous sections. Specifically, I assume that

$$v < x^* + \mu, \tag{15}$$

where x^* is the efficient level of consumption in DM transactions that satisfies $u'(x^*) = 1$. This assumption implies that consumption in DM transactions for buyers choosing bank contracts is inefficiently low due to a shortage of collateral. Moreover, it also implies that the central bank cannot support the efficient level of consumption for buyers opting out of banking contracts. This is because the quantity of currency issued by the central bank is constrained by the size of its balance sheet, which can only expand through open market purchases.

Given this assumption, the following proposition characterizes the effective lower bound (ELB) on nominal interest rates R^m.

[1] For cases with a sufficiently high cost of theft, equilibrium conditions can be presented, but analyzing the effects of monetary policy in such cases may not be straightforward.

Proposition C.1 (Effective Lower Bound) *Suppose that both ε and μ are sufficiently low, and that v satisfies (15). Then, an equilibrium exists if and only if*

$$R^m \geq \frac{u'(x^o)}{\eta[(1-\delta)u'(x^d)+\delta]}, \tag{16}$$

where (x^o, x^d), together with x^c, consist of the solution to $(x^o + \beta\mu)u'(x^o) = v$, $u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta$, and $U^b = U^o$. Furthermore,

$$\frac{u'(x^o)}{(1-\delta)u'(x^d)+\delta} \geq 1.$$

Interestingly, if v (the real value of consolidated government debt outstanding) is sufficiently low, equilibrium exists only if the nominal interest rate R^m is sufficiently high to support banking activities. In general, a fall in R^m increases the demand for currency both intensively and extensively, forcing the central bank to issue more currency through open market purchases. However, in a complete disintermediation scenario (i.e., $\theta = 0$), the central bank would be unable to meet public demand for currency, as the scale of its open market purchases is limited by the supply of government bonds. Thus, complete disintermediation cannot be sustained in equilibrium, creating an additional lower bound on R^m.[2]

Another notable finding is that the ELB on nominal interest rates can even be positive. This occurs due to the shortage of collateralizable assets, which creates inefficiencies in the banking system. When the nominal interest rate is zero ($R^m = 1$), each contracting buyer would consume the same quantity of goods across different DM meetings, i.e., $x^c = x^d$. However, if the fixed cost of holding currency μ is sufficiently low, a non-contracting buyer would consume more in the DM than a contracting buyer, i.e., $x^o > x^c = x^d$, leading to complete disintermediation ($\theta = 0$). Since this outcome is not sustainable in equilibrium,

[2]This finding is related to Eggertsson, Juelsrud, Summers, and Wold (2022), who show the coexistence of the lower bound on deposit rates (disintermediation-free interest rates) and the lower bound on short-term policy rates (arbitrage-free interest rates).

the nominal interest rate must be strictly positive to support the banking system.[3]

This result implies that, given a sufficiently low μ, the nominal interest rates that support the banking system are sufficiently high to prevent arbitrage opportunities from storing currency across periods. Thus, the ELB is determined by the lower bound on disintermediation-free interest rates (the right-hand side of 16) rather than the lower bound on arbitrage-free interest rates.[4] From (16), introducing a non-par exchange rate η is still effective in lowering the ELB. However, the proportional cost of storing currency γ becomes irrelevant because storing currency across periods is never profitable.

C.3 Characterization of Equilibrium

Here, I focus on cases where $\mu = 0$ (no fixed cost of holding currency at the beginning of the TS) for analytical convenience.[5] The equilibrium conditions are then given by

$$\eta R^m = \frac{u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \tag{17}$$

$$(1-\rho)\theta x^d \left[u'(x^d) + \frac{\delta}{1-\delta}\right] + \rho\theta(x^c)\left[u'(x^c) + \frac{\delta}{1-\delta}\right] + (1-\theta)(x^o)u'(x^o) = v, \tag{18}$$

$$\pi = \frac{\beta\left[u'(x^c) - \delta u'(x^d) + \delta\right]}{\eta}. \tag{19}$$

Equations (17)-(19) come from the first-order conditions of a private bank's problem in equilibrium, with (18) representing the collateral constraint. Notably, this collateral constraint differs from that in the baseline model (P35), as it accounts for the central bank's purchases

[3]One can imagine that a higher fixed storage cost μ would increase inefficiencies in currency-based DM transactions. This would discourage buyers from opting out of deposit contracts, thereby lowering the ELB. While analyzing the qualitative effect of an increase in μ appears to be complicated, the intuition suggests that a sufficiently high μ would lead to a negative ELB on nominal interest rates.

[4]This finding is related to Eggertsson, Juelsrud, Summers, and Wold (2022), in that the lower bound on deposit rates (disintermediation-free interest rates) tends to be higher than the lower bound on short-term policy rates (arbitrage-free interest rates). While my model abstracts from addressing the conflict between these different lower bounds, it suggests that introducing a non-par exchange rate policy could reduce both lower bounds by decreasing the rate of return on holding currency for both banks and depositors.

[5]This assumption simplifies the analysis while still providing insights that can be extended to cases with sufficiently low $\mu > 0$.

of government bonds when issuing currency for non-contracting buyers.

From each buyer's problem in the CM, represented by (8) and (10)-(12), I obtain

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \tag{20}$$

$$\rho\left[u(x^c) - x^c u'(x^c)\right] + (1-\rho)\left[u(x^d) - x^d u'(x^d)\right] \geq u(x^o) - x^o u'(x^o). \tag{21}$$

Equation (20) ensures positive consumption quantities in the DM for both contracting and non-contracting buyers while also requiring that their choices be incentive-compatible in equilibrium. Since collateral scarcity constrains both x^c and x^d, buyers opt for deposit contracts only when $x^c < x^d$, as they would otherwise prefer to opt out. Moreover, if $x^o < x^c < x^d$, no buyers would opt out of deposit contracts, whereas if $x^c < x^d < x^o$, no buyers would opt in. Thus, the DM consumption quantities must satisfy $x^c < x^o < x^d$ in equilibrium, as indicated by (20).[6] Equation (21) further ensures that buyers weakly prefer opting for a deposit contract over opting out.

Finally, from (13)-(14), incentive-compatibility conditions in the TS are given by

$$\alpha^b = \frac{\eta - 1}{\eta}, \tag{22}$$

$$\alpha^s = \frac{\beta\varepsilon}{\eta\left[\theta\rho x^c + (1-\theta)x^o\right]}, \tag{23}$$

$$\varepsilon < \frac{\eta\left[\theta\rho x^c + (1-\theta)x^o\right]}{\beta}. \tag{24}$$

Equations (22) and (23) determine the fraction of buyers acquiring theft technology α^b and the fraction of sellers carrying currency into the TS α^s. Inequality (24) is a necessary condition for this equilibrium to exist.

The model can be solved differently depending on whether (21) holds with equality. If (21) holds with equality, equations (17), (20), and (21) determine x^c, x^d, and x^o given

[6] However, using only currency in the DM is also inefficient due to the fixed cost of holding currency μ. With a sufficiently high μ, buyers could choose to use deposit contracts even when $x^d \leq x^c$.

monetary policy (R^m, η) and fiscal policy v. Then, equation (18) solves for θ, equation (19) solves for π, and equations (22) and (23) solve for α^b and α^s, respectively.

On the other hand, if (21) holds with strict inequality, equations (17) and (18), with $\theta = 1$, solve for x^c and x^d. Then, equation (19) determines π, while (22) and (23) solve for α^b and α^s, respectively. Equation (20) solves for x^o, which is the would-be consumption quantity in the DM if a buyer were to opt out of banking arrangements off equilibrium.

C.4 Effects of Monetary Policies

The following proposition shows how the fraction θ is determined in equilibrium and how monetary policy (R^m, η) affects the equilibrium outcome depending on the value of θ.

Proposition C.2 (Comparative Statics) *Suppose $\mu = 0$. If ηR^m is sufficiently high to satisfy (16) but not too high, then $0 \leq \theta < 1$ in equilibrium and (21) holds with equality. In this case, $\frac{dx^c}{d(\eta R^m)} < 0$, $\frac{dx^o}{d(\eta R^m)} < 0$, $\frac{dx^d}{d(\eta R^m)} > 0$, $\frac{dr}{d(\eta R^m)} > 0$, and $\frac{d\theta}{d(\eta R^m)} > 0$. Furthermore, $\frac{d\pi}{dR^m} > 0$, $\frac{d\alpha^b}{dR^m} = 0$, $\frac{d\alpha^s}{dR^m} > 0$, $\frac{d\pi}{d\eta} < 0$, and $\frac{d\alpha^b}{d\eta} > 0$, while $\frac{d\alpha^s}{d\eta}$ is ambiguous. If ηR^m is very high, then $\theta = 1$ in equilibrium and (21) holds with strict inequality.*

With the non-par exchange rate η held constant, an increase in the nominal interest rate R^m decreases the rate of return on currency relative to reserves and government bonds. This leads to a substitution of deposit claims for currency. If $0 \leq \theta < 1$ in equilibrium, the consumption quantity in DM transactions using deposit claims x^d increases along with a rise in the fraction of contracting buyers θ, while the consumption quantities in DM transactions using currency x^c and x^o decrease. Thus, lowering R^m contributes to disintermediation by incentivizing more buyers to opt out of banking arrangements, i.e., by decreasing θ.

The central bank can offset this effect by increasing the non-par exchange rate η, as this helps maintain the relative rate of return on currency $1/\eta R^m$ at the previous level. If ηR^m is held constant, the fraction θ as well as the consumption quantities x^c, x^d, and x^o would remain unchanged. As a result, a non-par exchange rate enables the central bank to implement negative nominal interest rates without triggering disintermediation. However, this policy leads to welfare losses, similar to the baseline model, as it increases the fraction of buyers investing in the costly theft technology α^b.

C.5 Optimal Monetary Policy

The welfare measure for this economy is defined as

$$\mathcal{W} = \rho\theta\left[u\left(x^c\right) - x^c\right] + (1-\rho)\,\theta[u(x^d) - x^d] + (1-\theta)\left[u\left(x^o\right) - x^o\right] - \alpha^b\varepsilon. \tag{25}$$

which is the sum of surpluses from trade in the CM and the DM, net of the total cost incurred in the TS. Then, the following proposition characterizes the optimal monetary policy.

Proposition C.3 (Optimal Monetary Policy) *Suppose $\mu = 0$. Then, the optimal monetary policy consists of $\eta = 1$ and $R^m = \frac{u'(x^o)}{u'(x^d)-\delta u'(x^d)+\delta} > 1$, where (x^o, x^d), together with x^c, consist of the solution to $x^o u'(x^o) = v$, (20), and (21) with equality.*

The central bank can optimize the surplus from trade by choosing an appropriate policy combination of (R^m, η). While there exists a set of such combinations that achieve the maximized level of trade surplus, maintaining the one-to-one exchange rate between currency and reserves ($\eta = 1$), as seen in a traditional central banking system, is always optimal. This is because the traditional one-to-one exchange rate enables the central bank to eliminate costly theft without reducing welfare. Moreover, optimality is achieved when the central bank sets the nominal interest rate on reserves R^m at the base lower bound (the ELB level under $\eta = 1$). Given that the base lower bound on R^m is above zero, the consumption quantity in deposit-based DM transactions exceeds those in currency-based transactions for any R^m above the base lower bound. Consequently, reducing R^m until it reaches the lower bound maximizes welfare as it allows consumption smoothing across DM transactions.

C.6 Discussion

When considering real-world payment systems, the baseline model and the extended model capture different aspects. The baseline model effectively reflects the distinction between cash transactions and electronic transactions, where some transactions can only be made with currency due to various reasons such as accessibility or privacy concerns, while others solely involve deposit claims, as seen in online transactions. In contrast, the extended model addresses the substitutability between currency and deposit claims but does not consider electronic transactions where only deposit claims are accepted. Therefore, neither model fully captures the complexity and diversity of real-world payment systems, but both provide valuable insights into the implications of implementing a non-par exchange rate policy for the ELB and welfare.

D Quantitative Analysis

Theoretically, introducing a non-par exchange rate between currency and reserves can lead to welfare losses by incentivizing costly theft and distorting the equilibrium allocation. To quantify these welfare losses, I calibrate the baseline model to the U.S. economy and conduct a counterfactual analysis to evaluate the effects of introducing a non-par exchange rate between currency and reserves.

D.1 Calibration

I consider an annual model where the utility function in the DM takes the form $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$. When calibrating the baseline model to data, I exclude the cost of theft ε because a one-to-one exchange rate between currency and reserves implies no theft occurring in the model.[7] Then, there are eight parameters to calibrate: σ (the curvature of DM consumption), β

[7] To quantify the welfare cost arising from an increase in theft, the cost of theft parameter ε needs to be calibrated outside the model. However, to the best of my knowledge, there is no available data that allows for the direct measurement of this parameter.

Parameters	Values	Calibration targets	Sources
β	0.96	Standard in literature	
R^m	1.0025	Avg. interest rate on reserves: 0.25%	FRED
γ	0.00	Lowest target range for fed funds rate: 0-0.25%	FRED
σ	0.17	Money demand elasticity (1959-2007): -4.19	FRED
ρ	0.17	Currency to M1 ratio: 17.22%	FRED; Lucas and Nicolini (2015)
v	1.13	Avg. locally-held public debt to GDP: 66.73%	FRED
δ	0.45	Avg. inflation rate: 1.06%	FRED
μ	0.01	Fixed storage cost: 2% of currency payments	Author's assumption

Table 1: Calibration results

(the discount factor), ρ (the fraction of currency transactions in the DM), μ (the fixed cost of storing currency), γ (the proportional cost of storing currency), δ (the fraction of assets private banks can abscond with), R^m (the nominal interest rate on reserves), and v (the value of the consolidated government's liabilities held by the public).

Table 1 summarizes the calibration results along with the target moments, which are mostly constructed from U.S. data for 2013-2015. This period is chosen because, for the purpose of this exercise, it is suitable to consider a timeframe when the policy rate was close to zero. Also, key variables such as the nominal interest rate on reserves and domestically-held public debt to GDP were stable during this period.

There are three parameters calibrated externally. The discount factor β is given by $\beta = 0.96$. From Federal Reserve Economic Data (FRED), the nominal interest rate on reserves was 0.25 percent over the period 2013-2015 ($R^m = 1.0025$). Finally, the lowest target range for the federal funds rate has been between 0 and 0.25 percent since 1954. Although this does not imply that the proportional cost of storing currency is zero, the proportional cost γ is assumed to be zero for convenience.[8]

Calibrating σ (the curvature of DM consumption) involves matching the elasticity of money demand in the model with the empirical money demand elasticity obtained from the

[8]The proportional cost of storing currency implies that the ELB on the nominal interest rate can be negative. However, the Federal Reserve might have faced legal and political issues with implementing negative nominal interest rates. As negative rates have not been explored in the U.S., it seems difficult to calibrate the proportional cost of storing currency with this model.

data. Estimating the money demand elasticity requires a longer time series of data, so I select the time period from 1959 to 2007.[9] Using data on currency in circulation and nominal GDP (from FRED), I calculate the currency-to-GDP ratios. Then, the money demand elasticity can be estimated using Moody's AAA corporate bond yields (from FRED) and the currency-to-GDP ratios, resulting in an estimated elasticity of -4.19.[10]

Then, I jointly calibrate four parameters: the curvature of DM consumption σ, the fraction of currency transactions in the DM ρ, the value of government liabilities held by the public v, the fraction of bank assets that can be absconded δ, and the fixed cost of storing currency μ. The curvature parameter σ is calibrated to match the estimated money demand elasticity. Using currency-in-circulation data from FRED and the new M1 series from Lucas and Nicolini (2015), I calibrate the fraction of currency transactions in the DM ρ until the model generates the observed currency-to-M1 ratio. I use domestically-held public debt to GDP from FRED to calibrate the value of publicly-held government liabilities v.[11] Another variable I use to calibrate parameters is the inflation rate. Along with other parameters, the fraction of assets that can be absconded δ is calibrated so that the model generates an inflation rate consistent with the observed rate of 1.06 percent. Finally, I set the fixed cost of storing currency μ to be 2 percent of cash payments.[12]

[9]In the aftermath of the global financial crisis in 2007-2008, the demand for currency increased possibly due to non-transactional purposes. To calculate the elasticity of money demand specifically for transactions, I exclude post-crisis data, following the approach of Chiu, Davoodalhosseini, Jiang, and Zhu (2022) and Altermatt (2022).

[10]The interest rate on liquid bonds (e.g., the 3-month Treasury Bill rate) may fluctuate due to changes in the liquidity premium. To exclude such a possibility, I consider the AAA corporate bond yield as the nominal interest rate on illiquid bonds and $\frac{\pi}{\beta} - 1$ as its theoretical counterpart.

[11]I define domestically-held public debt as the total public debt net of public debt held by foreign and international investors.

[12]Under this assumption, each buyer pays approximately 2 percent more to purchase goods in currency transactions, compensating for the seller's storage cost. While the fixed storage cost may indeed deviate from 2 percent, adjusting it within the range of 0 percent to 10 percent does not significantly affect the outcomes of the counterfactual analysis.

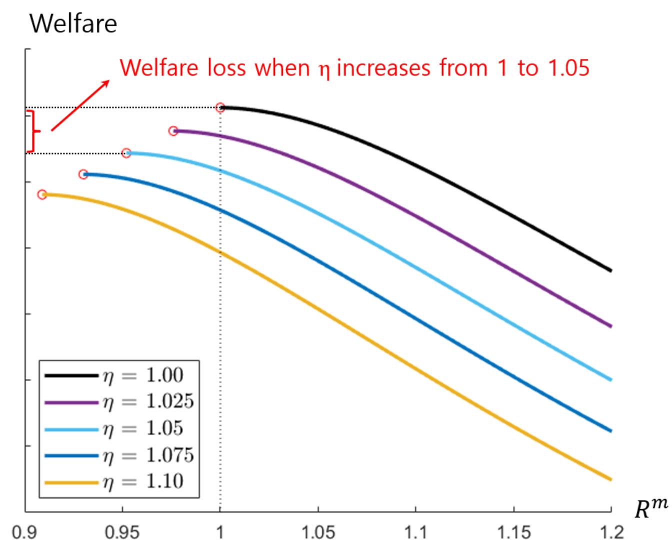

Figure 1: Monetary policy (R^m, η) and welfare

D.2 Counterfactual Analysis

I consider three different scenarios where the fixed cost of theft ε is set to (i) 2.5 percent, (ii) 5 percent, and (iii) 10 percent of the current consumption level. With the calibrated parameters and each value of ε, I vary the non-par exchange rate η and compute the corresponding nominal interest rate on reserves that maximizes welfare, denoted by R^*_η. As depicted in Figure 1, an increase in η reduces the ELB on the nominal interest rate. However, the welfare level under the optimal nominal interest rate R^*_η decreases as η rises.

My approach to quantifying the welfare cost of increasing η involves measuring how much consumption individuals would need to be compensated to tolerate the non-par exchange rate η. Suppose that, for any given η, the central bank sets the nominal interest rate at the optimal level, i.e., $R^m = R^*_\eta$. Then, the welfare measure can be expressed as

$$\mathcal{W}(R^m = R^*_\eta, \eta) = \rho[u(x^c) - x^c] + (1-\rho)[u(x^d) - x^d] - \alpha^b \varepsilon.$$

If the consumption quantities in the DM are adjusted by a factor Δ, the adjusted level of

η	ELB	R^*_η	$(\Delta_\eta - 1) \times 100$		
			$\varepsilon = 2.5\%$	$\varepsilon = 5\%$	$\varepsilon = 10\%$
1.00	1.000	1.000	-	-	-
1.025	0.976	0.976	0.0561	0.1118	0.2236
1.05	0.952	0.952	0.1095	0.2183	0.4367
1.075	0.930	0.930	0.1604	0.3199	0.6399
1.10	0.909	0.909	0.2091	0.4168	0.8339

Table 2: ELB, optimal interest rate, and the welfare cost of reducing the ELB

welfare can be expressed as

$$\mathcal{W}_\Delta(R^m = R^*_\eta, \eta) = \rho[u(\Delta x^c) - x^c] + (1-\rho)[u(\Delta x^d) - x^d] - \alpha^b \varepsilon.$$

Then, I compute the value Δ_η that solves $\mathcal{W}_{\Delta_\eta}(R^m = R^*_\eta, \eta > 1) = \mathcal{W}(R^m = R^*_\eta, \eta = 1)$. The welfare cost of introducing η can be measured as $\Delta_\eta - 1$ percent of consumption. If individuals are compensated with this amount of consumption, they would be indifferent between the two policy choices: a one-to-one exchange rate and a non-par exchange rate.

Table 2 presents the ELB, the optimal nominal interest rate R^*_η, and the welfare cost of introducing a non-par exchange rate η for various fixed costs of theft ε. Notably, an increase in η reduces both the ELB and the optimal interest rate, regardless of the cost of theft. Recall that, with a fixed cost of storing currency close to zero ($\mu \approx 0$), the optimal monetary policy can be characterized by a modified Friedman rule ($\eta R^m \approx 1$). Thus, an increase in η leads to a decrease in the optimal nominal interest rate R^*_η. Assuming that monetary policy is conducted optimally for any given non-par exchange rate η, there would be no distortion in the equilibrium prices and allocations.

However, introducing a non-par exchange rate η increases the aggregate cost of theft in equilibrium. For instance, if the fixed cost of theft ε is 2.5 percent of the current consumption level, increasing η by 5 percent and 10 percent would cost, respectively, 0.11 percent and 0.21 percent of consumption. If ε is 5 percent of the current consumption level, the corresponding increases in η would cost 0.22 percent and 0.42 percent of consumption. Finally, with ε set at 10 percent of the current consumption level, the increases in η by 5 percent and 10 percent would cost 0.44 percent and 0.84 percent of consumption, respectively.[13]

The welfare cost of introducing a non-par exchange rate can be better understood by comparing it with estimates for the welfare cost of another policy frequently discussed in the literature: the welfare cost of 10 percent inflation. Estimates for the welfare cost of 10 percent inflation are typically around 1 percent of consumption. Therefore, the welfare cost of introducing a non-par exchange rate seems significant.[14] It is important to note that the welfare cost of using a non-par exchange rate critically depends on the fixed cost of investing in the theft technology ε. As ε increases, the welfare cost also increases proportionally.

E Omitted Proofs

Proof of Lemma 1: First, consider the case where $\eta = 1$. If theft does not occur in equilibrium ($\alpha^b = 0$), sellers would be indifferent between depositing their currency with the central bank and engaging in side-trading with private banks ($\alpha^s \in [0, 1]$). For this equilibrium to be incentive-compatible, the cost of theft ε must be sufficiently high to satisfy $\varepsilon \geq \frac{\rho\alpha^s\eta c'}{\pi}$. Moreover, if $\varepsilon > \frac{\rho\eta c'}{\pi}$, investing in theft ($\alpha^b = 0$) is never optimal for buyers for any $\alpha^s \in [0, 1]$. If $\varepsilon = \frac{\rho\eta c'}{\pi}$, buyers become indifferent between investing in theft and not investing, although only $\alpha^b = 0$ can be sustainable in equilibrium. Conversely, if $\varepsilon < \frac{\rho\eta c'}{\pi}$, the fraction of sellers carrying currency into the TS must be sufficiently low to prevent theft. For ε to be sufficiently high to satisfy $\varepsilon \geq \frac{\rho\alpha^s\eta c'}{\pi}$, the fraction α^s must fall within the range of $[0, \bar{\alpha}^s]$, where $\bar{\alpha}^s = \frac{\varepsilon\pi}{\rho\eta c'}$. Therefore, when $\eta = 1$, there exist a continuum of no-theft equilibria with $\alpha^b = 0$ and $\alpha^s \in [0, 1]$ for $\varepsilon \geq \rho\eta c^s$, and with $\alpha^b = 0$ and $\alpha^s \in [0, \bar{\alpha}^s]$ for $\varepsilon < \frac{\rho\eta c'}{\pi}$, where $\bar{\alpha}^s = \frac{\varepsilon\pi}{\rho\eta c'}$. However, a theft equilibrium does not exist in this case. This is because,

[13] Using different values for the fixed cost of storing currency μ does not significantly change the result. For example, if μ is 10 percent of cash payments and ε is 5 percent of the current consumption level, increasing η by 5 percent and 10 percent would cost, respectively, 0.2185 percent and 0.4172 percent of consumption.

[14] For instance, the welfare cost of increasing inflation from 0 percent to 10 percent is reported as 0.62 percent of consumption in Chiu and Molico (2010), 0.87 percent in Lucas (2000), and 1.32 percent (assuming that buyers make a take-it-or-leave-it offer to sellers) in Lagos and Wright (2005), among other studies.

when buyers steal currency ($\alpha^b \in (0, 1]$), no sellers would carry their currency into the TS ($\alpha^s = 0$), which disincentivizes buyers from investing in the costly theft technology.

Next, I consider the case where $\eta > 1$. If no theft occurs in equilibrium ($\alpha^b = 0$), sellers would prefer side-trading their currency with private banks over depositing it with the central bank ($\alpha^s = 1$). A necessary condition for this equilibrium to exist is $\varepsilon \geq \frac{\rho\eta c'}{\pi}$. Therefore, a unique equilibrium exists with $\alpha^b = 0$ and $\alpha^s = 1$ for $\varepsilon \geq \frac{\rho\eta c'}{\pi}$. If $\varepsilon < \frac{\rho\eta c'}{\pi}$, buyers would choose to incur costs to acquire theft technology ($\alpha^b = 1$). However, cases with $\alpha^b = 1$ (where buyers strictly prefer to invest in theft) cannot be sustained in equilibrium because $\alpha^b = 1$ would lead to $\alpha^s = 0$, discouraging costly theft. Thus, in equilibrium, both buyers and sellers must be indifferent between their options. From (P7) and (P8), I obtain:

$$\alpha^b = \frac{\eta - 1}{\eta}, \tag{26}$$

$$\alpha^s = \frac{\varepsilon\pi}{\rho\eta c'}. \tag{27}$$

Hence, for $\varepsilon < \frac{\rho\eta c'}{\pi}$, there exists a unique equilibrium that satisfies the above equations.

Proof of Proposition 1: In equilibrium, equations (P27) and (P29) solve for x^c and x^d. For the comparative statics analysis with respect to R^m, I totally differentiate (P27) and (P29) and evaluate the derivatives of x^c and x^d for $(R^m, \eta) = (1, 1)$ and $\mu = 0$, which yield:

$$\frac{dx^c}{dR^m} = \frac{(1-\rho)[(1-\delta)u'(x) + \delta][(1-\delta)(1-\sigma)u'(x) + \delta]}{u''(x)[(1-\delta)(1-\sigma)u'(x) + \delta + \rho\mu(1-\delta)u''(x)]} < 0,$$

$$\frac{dx^d}{dR^m} = \frac{-\rho[(1-\delta)u'(x) + \delta][(1-\delta)(1-\sigma)u'(x) + \delta + \mu(1-\delta)u''(x)]}{u''(x)[(1-\delta)(1-\sigma)u'(x) + \delta + \rho\mu(1-\delta)u''(x)]} > 0,$$

where $x^c = x^d = x$ and $\sigma \equiv -\frac{xu''(x)}{u'(x)} \in (0, 1)$. It follows immediately that from (P26) and (P28), r^m, r^b, and π increase, while from the first argument in (P31), the ELB remains unchanged.

Now, I turn attention to the comparative statics analysis with respect to η. Then, evaluating the derivatives of x^c and x^d with respect to η for $(R^m, \eta) = (1, 1)$ and $\mu = 0$ yields:

$$\frac{dx^c}{d\eta} = -\frac{\delta\{\rho\sigma u'(x) + (1-\rho)[(1-\delta)(1-\sigma)u'(x) + \delta][u'(x) - 1] - \mu\rho u''(x)\}}{u''(x)[(1-\delta)(1-\sigma)u'(x) + \delta + \rho\mu(1-\delta)u''(x)]} > 0,$$
$$\frac{dx^d}{d\eta} = \frac{\rho\delta[(1-\sigma)u'(x) - 1 + \mu u''(x)][(1-\delta)u'(x) + \delta]}{u''(x)[(1-\delta)(1-\sigma)u'(x) + \delta + \rho\mu(1-\delta)u''(x)]}.$$

Additionally, evaluating equation (P29) for $(R^m, \eta) = (1, 1)$ and $\mu = 0$ yields:

$$\left[u'(x) + \frac{\delta}{1-\delta}\right](x + \rho\mu) = v, \tag{28}$$

where x is increasing in v. The collateral constraint (28) does not bind in equilibrium if

$$v \geq \frac{x^* + \rho\mu}{1-\delta}, \tag{29}$$

where x^* is the first-best quantity of consumption in the DM that satisfies $u'(x^*) = 1$. Let $\bar{v}$ denote the right-hand side of inequality (29), $\hat{x}$ denote the solution to $u'(x) = \frac{1}{1-\sigma}$, and $\hat{v}$ denote the solution to (28) when $x = \hat{x}$. Then, the derivatives of x^d with respect to η can be written as

$$\frac{dx^d}{d\eta} \leq 0, \qquad \text{if} \quad v \in (0, \hat{v}]$$
$$\frac{dx^d}{d\eta} > 0. \qquad \text{if} \quad v \in (\hat{v}, \bar{v})$$

Thus, from (P26), an increase in η decreases r^m and r^b for $v \in (0, \hat{v}]$ and increases r^m and r^b for $v \in (\hat{v}, \bar{v})$. From (P28) or

$$\pi = \beta R^m \left[u'(x^d) - \delta u'(x^d) + \delta\right],$$

an increase in η increases π for $v \in (0, \hat{v}]$ and decreases π for $v \in (\hat{v}, \bar{v})$. Finally, from the

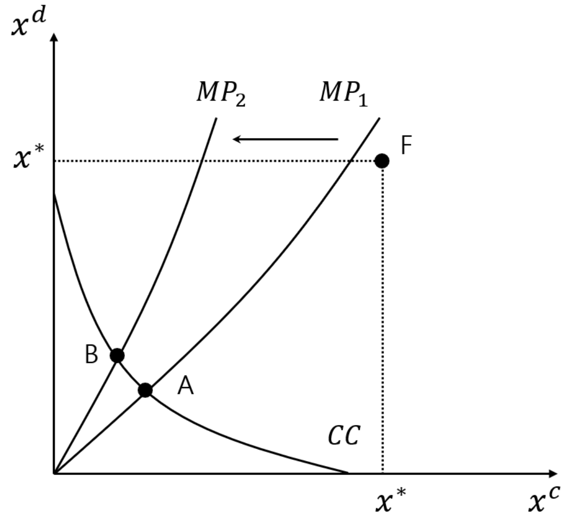

Figure 2: Effects of monetary policy (R^m, η)

first argument in (P31), the ELB falls.

Proof of Proposition 2: The DM consumption quantities, x^c and x^d, are determined by equations (P33) and (P35). Equation (P35) can be expressed as

$$F(x^c, x^d) = v, \tag{30}$$

and I can show that the function $F(\cdot, \cdot)$ is strictly increasing in both $0 \leq x^c < x^*$ and $0 \leq x^d < x^*$, as $-x\frac{u''(x)}{u'(x)} < 1$. Again, x^* denotes the first-best quantity of consumption in the DM that satisfies $u'(x^*) = 1$. Based on this property, equation (30) can be depicted by a downward-sloping locus in the (x^c, x^d) space, given v. Conversely, equation (P33) can be depicted by an upward-sloping locus in the (x^c, x^d) space, given (R^m, η).

For the comparative statics analysis, suppose the central bank raises R^m while holding η constant. As illustrated in Figure 2, the MP curve, representing equation (P33), shifts to the left while the CC curve, representing equation (30), remains unaffected. Thus, x^c decreases, and x^d increases. Then, from (P32) and (P34), π rises, and real interest rates (r^m, r^b) rise. From (P36) and (P38), α^s increases, while α^b and the ELB remain unchanged.

Next, suppose that the central bank raises η while holding R^m constant. This monetary intervention also shifts the MP curve leftward, while leaving the CC curve unaffected. Consequently, x^c decreases, and x^d increases. Then, from (P34), real interest rates (r^m, r^b) rise. Using (P33), equation (P32) can be written as:

$$\pi = \beta R^m \left[u'(x^d) - \delta u'(x^d) + \delta\right],$$

so π falls. From (P36) and (P38), α^b increases, and the ELB decreases. However, the effect on α^s is ambiguous since ηx^c can increase or decrease depending on parameters.

Proof of Proposition 3: The proof consists of two steps. First, I will identify monetary policies that maximize the welfare measure $\mathcal{W}$, assuming α^b is exogenously given. Then, I will determine the optimal monetary policy considering α^b as endogenously determined in response to monetary policy.

In the first step, I solve the following maximization problem given $\alpha^b \in [0,1]$:

$$\max_{(R^m,\eta)} \rho\left[u\left(x^c\right) - x^c\right] + (1-\rho)\left[u\left(x^d\right) - x^d\right] - \alpha^b\varepsilon \tag{31}$$

subject to

$$\eta R^m = \frac{u'\left(x^c\right) - \delta u'\left(x^d\right) + \delta}{u'\left(x^d\right) - \delta u'\left(x^d\right) + \delta}, \tag{32}$$

$$\left[u'(x^c) + \frac{\delta}{1-\delta}\right]\rho(x^c + \mu) + \left[u'\left(x^d\right) + \frac{\delta}{1-\delta}\right](1-\rho)x^d = v, \tag{33}$$

$$R^m \geq \frac{1}{\eta + \beta\gamma}, \quad \eta \geq 1 \tag{34}$$

Here, a monetary policy measure relevant to welfare in equilibrium is ηR^m, denoted as $\Omega \equiv \eta R^m$. Hence, from (34), Ω must satisfy $\Omega \geq \frac{\eta}{\eta+\beta\gamma}$. Differentiating the objective (31) with respect to Ω yields:

$$\frac{d\mathcal{W}}{d\Omega} = \rho\left[u'(x^c) - 1\right]\frac{dx^c}{d\Omega} + (1-\rho)\left[u'(x^d) - 1\right]\frac{dx^d}{d\Omega}. \tag{35}$$

Let $\sigma = -\frac{xu''(x)}{u'(x)}$. Then, from total differentiation of (32) and (33), I derive:

$$\frac{dx^c}{d\Omega} = \frac{(1-\rho)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Phi} < 0, \tag{36}$$

$$\frac{dx^d}{d\Omega} = \frac{-\rho\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta} + \mu u''(x^c)\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Phi} > 0, \tag{37}$$

where

$$\Phi = (1-\rho)u''(x^c)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] \\ + \rho u''(x^d)\left[(1-\delta)\Omega + \delta\right]\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta} + \mu u''(x^c)\right] < 0,$$

for sufficiently low μ. A monetary policy Ω attains a local optimum if the resulting consumption quantities x^c and x^d lead to $\frac{d\mathcal{W}}{d\Omega} = 0$. From (35)-(37), I can characterize the optimal allocation x^c and x^d as follows:

$$\begin{aligned}
&\frac{d\mathcal{W}}{d\Omega} = 0,\\
&\Leftrightarrow \left[u'(x^c) - 1\right]\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] - \left[u'(x^d) - 1\right]\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta} + \mu u''(x^c)\right] = 0,\\
&\Rightarrow \left[u'(x^c) - 1\right]\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] \leq \left[u'(x^d) - 1\right]\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta}\right],\\
&\Leftrightarrow \frac{u'(x^c) - 1}{(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta}} \leq \frac{u'(x^d) - 1}{(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}}.
\end{aligned}$$

Since the function $F(x) = \frac{u'(x)-1}{(1-\sigma)u'(x)+\frac{\delta}{1-\delta}}$ is strictly decreasing in x, the above inequality is equivalent to $x^c \geq x^d$. Note that from (32) $x^c = x^d$ if $\Omega = 1$, and that as Ω rises, x^c decreases and x^d increases. Therefore, to achieve $x^c \geq x^d$, the optimal monetary policy Ω must satisfy

$\Omega \leq 1$, which holds with equality if and only if $\mu = 0$.

In the first step, I have shown that an optimal monetary policy must be a combination of (R^m, η) such that $\eta R^m \leq 1$. All optimal combinations of (R^m, η) lead to the same gains from trade in the DM, $\rho\left[u\left(x^c\right) - x^c\right] + (1-\rho)\left[u\left(x^d\right) - x^d\right]$. However, as the non-par exchange rate η rises, the fraction of buyers who choose to steal currency in the TS α^b increases (P36), leading to a increase in the total cost of theft in the TS. Therefore, the optimal monetary policy is given by $\eta = 1$ and $R^m \leq 1$.

Even if the optimal R^m is constrained by (34), the optimal interest rate is the lower bound $\frac{1}{\eta+\beta\gamma}$. An increase in η allows R^m to decrease further, but it acts to increase the current level of Ω. So, an increase in η cannot increase the sum of surpluses from trade in the CM and the DM, represented by the first two terms in the welfare measure (31), while increasing the total theft cost in the TS.

Proof of Proposition 4: Suppose the cost of theft ε is sufficiently high to prevent theft in equilibrium ($\alpha^b = 0$). To show that social welfare increases with η, consider the scenario where the central bank sets the nominal interest rate R^m to achieve $x^c = x^d = x$ given a non-par exchange rate η. While this policy may not be optimal, it helps us understand the optimal level of η. When $x^c = x^d = x$, equations (P27) and (P29) can be written as

$$R^m = R^b = \frac{\eta u'(x) - \delta u'(x) + \delta}{\eta\left[u'(x) - \delta u'(x) + \delta\right]}, \tag{38}$$

$$u'(x)\left[x + \rho\mu\right] + \frac{\left[\rho + (1-\rho)\eta\right]\delta x + \rho\delta\mu}{(1-\delta)\eta} = v. \tag{39}$$

In this case, equation (39) solves for x, and then (38) solves for R^m given η. Furthermore, if the value of the consolidated government debt v is sufficiently low, or

$$v \leq \frac{\left[(1-\delta\rho)\eta + \delta\rho\right]x^* + \rho\mu\left[(1-\delta)\eta + \delta\right]}{(1-\delta)\eta},$$

then x increases with η for a sufficiently low μ.[15] Since the welfare measure can be written as $\mathcal{W} = u(x) - x$, an increase in η effectively increases welfare as long as the nominal interest rate R^m can be adjusted to achieve $x^c = x^d = x$.

However, according to Proposition 1, an increase in η must be accompanied by an increase in R^m to maintain the same consumption quantities in the two types of DM meetings, implying that the choice of R^m is not constrained by the ELB. Although the optimal R^m may not satisfy $x^c = x^d$, the fact that social welfare increases with η remains unchanged. Finally, η must be sufficiently low to prevent buyers from investing in theft technology. Therefore, at the optimum, η is chosen such that buyers are indifferent between stealing currency and not stealing.

Suppose further that there is no fixed cost of holding currency at the beginning of the TS, i.e., $\mu = 0$. Consider the following maximization problem given $\eta \geq 1$:

$$\max_{(R^m,\eta)} \rho \left[u\left(x^c\right) - x^c\right] + (1-\rho)\left[u\left(x^d\right) - x^d\right] \tag{40}$$

subject to

$$\eta R^m = \frac{\eta u'\left(x^c\right) - \delta u'\left(x^d\right) + \delta}{u'\left(x^d\right) - \delta u'\left(x^d\right) + \delta}, \tag{41}$$

$$\left[u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\rho x^c + \left[u'\left(x^d\right) + \frac{\delta}{1-\delta}\right](1-\rho)x^d = v, \tag{42}$$

$$R^m \geq \max\left\{\frac{1}{\eta + \beta\gamma}, \frac{\eta}{(\eta+\beta\gamma)[(1-\delta)u'(x^d) + \delta]}\right\}. \tag{43}$$

Differentiating the objective (40) with respect to R^m yields:

$$\frac{d\mathcal{W}}{dR^m} = \rho\left[u'(x^c) - 1\right]\frac{dx^c}{dR^m} + (1-\rho)\left[u'(x^d) - 1\right]\frac{dx^d}{dR^m}. \tag{44}$$

[15] Here, x^* is the solution to $u'(x) = 1$.

Letting $\sigma = -\frac{xu''(x)}{u'(x)}$ and totally differentiating (41) and (42) gives:

$$\frac{dx^c}{dR^m} = \frac{\eta(1-\rho)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Lambda} < 0,$$
$$\frac{dx^d}{dR^m} = \frac{-\eta\rho\left[(1-\sigma)u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Lambda} > 0,$$

where

$$\Lambda = (1-\rho)\eta u''(x^c)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] + \rho u''(x^d)\left[(1-\sigma)u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\left[\eta R^m(1-\delta) + \delta\right] < 0.$$

Then, I can evaluate the derivative of $\mathcal{W}$, or equation (44), for $\eta R^m = 1$. Noting that $\eta u'(x^c) = u'(x^d)$ from (41), I obtain

$$\left.\frac{d\mathcal{W}}{dR^m}\right|_{\eta R^m = 1} = \frac{\eta(1-\eta)\rho(1-\rho)\left[(1-\delta)u'(x^d) + \delta\right]}{\eta^2(1-\rho)u''(x^c) + \rho u''(x^d)} \geq 0, \tag{45}$$

implying that $\eta R^m \geq 1$ at an optimum.

Next, differentiating the objective (40) with respect to η gives:

$$\frac{d\mathcal{W}}{d\eta} = \rho\left[u'(x^c) - 1\right]\frac{dx^c}{d\eta} + (1-\rho)\left[u'(x^d) - 1\right]\frac{dx^d}{d\eta}. \tag{46}$$

I totally differentiate (41) and (42) to derive $\frac{dx^c}{d\eta}$ and $\frac{dx^d}{d\eta}$, and then evaluate the derivatives for $\eta R^m = 1$, which yields:

$$\frac{dx^c}{d\eta} = \frac{\delta\rho x^c u''(x^d) - \delta\eta(1-\rho)\left[u'(x^d) - 1\right]\left[(1-\delta)(1-\sigma)u'(x^d) + \delta\right]}{\eta\left[(1-\delta)(1-\sigma)u'(x^d) + \delta\right]\left[\rho u''(x^d) + \eta^2(1-\rho)u''(x^c)\right]} > 0,$$
$$\frac{dx^d}{d\eta} = \frac{\delta\rho\left\{\eta x^c u''(x^c) + \left[u'(x^d) - 1\right]\left[(1-\delta)(1-\sigma)u'(x^d) + \delta\right]\right\}}{\eta\left[(1-\delta)(1-\sigma)u'(x^d) + \delta\right]\left[\rho u''(x^d) + \eta^2(1-\rho)u''(x^c)\right]}.$$

Substituting the above expressions into (46) gives:

$$\left.\frac{d\mathcal{W}}{d\eta}\right|_{\eta R^m=1} = \frac{\mathit{\Gamma} + \rho\delta x^c \left\{\rho u''(x^d)\left[u'(x^c)-1\right] + \eta(1-\rho)u''(x^c)\left[u'(x^d)-1\right]\right\}}{\eta\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]\left[\rho u''(x^d)+\eta^2(1-\rho)u''(x^c)\right]}, \tag{47}$$

where

$$\mathit{\Gamma} = \rho\delta(1-\rho)(\eta-1)\left[u'(x^d)-1\right]\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right] \geq 0.$$

From (47), the derivative of $\mathcal{W}$ is strictly positive if $\eta = R^m = 1$, i.e., $\left.\frac{d\mathcal{W}}{d\eta}\right|_{\eta=R^m=1} > 0$. This implies that the monetary policy at $\eta = R^m = 1$ is not optimal. Therefore, from (45) and (47), I conclude that the optimal monetary policy is characterized by $\eta R^m > 1$.

Proof of Proposition C.1: Suppose that $\theta = 0$ in equilibrium. From equations (1)-(3) and (8),

$$\eta R^m \left[u'(x^d) - \delta u'(x^d) + \delta\right] = u'(x^o), \tag{48}$$

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \tag{49}$$

where (x^c, x^d) are the off-equilibrium consumption quantities in the DM, if a buyer were to participate in banking contracts. It can be shown that $\left|\frac{d[u'(x^d)-\delta u'(x^d)+\delta]}{d[\eta R^m]}\right| < 1$, so from (48) x^o increases with a decrease in ηR^m. However, the limited quantity of collateral (represented by $v < x^* + \mu$) implies that, from (18), the highest possible quantity for x^o is $\bar{x}$ that solves $(\bar{x} + \mu)u'(\bar{x}) = v$, and $\bar{x} < x^*$. So, any ηR^m that leads to x^o higher than $\bar{x}$ cannot be supported in equilibrium, implying that, from (48),

$$R^m \geq \frac{u'(\bar{x})}{\eta\left[u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta\right]}, \tag{50}$$

where $\underline{x}^d$ is the off-equilibrium consumption quantity in deposit-based DM transactions that is consistent with (49). Also, any R^m higher than the lower bound in (50) results in x^o such that $(x^o + \beta\mu)u'(x^o) < v$. This implies that $0 < \theta \leq 1$ and $U^b \geq U^o$ in equilibrium. So, by continuity, the off-equilibrium consumption quantities (x^c, x^d) when $R^m = \frac{u'(\bar{x})}{\eta\left[u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta\right]}$ must satisfy (49) and $U^b = U^o$ from (10)-(11) given $x^o = \bar{x}$.

Recall that the nominal interest rate R^m must also satisfy (P38), the no-arbitrage condition for agents from holding currency across periods. To prove that the lower bound on R^m given in (50) is always higher than the lower bound in (P38), I claim that the following condition must hold in equilibrium if the fixed cost of holding currency μ is close to zero:

$$\frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta} \geq 1. \tag{51}$$

To prove this claim, suppose $\eta R^m = 1$. In this case, deposit contracts allow buyers to consume the same quantity of goods across two types of DM transactions, i.e., $x^c = x^d = x$. Then, the quantity of DM consumption for buyers opting out of deposit contracts is higher than the quantity for those holding deposit contracts $(x^o > x)$ because, from (48)-(49),

$$u'(x^o) = (1 - \delta)u'(x) + \delta.$$

This implies that the expected utility for buyers opting out of contracts is higher than the expected utility for those opting in, because from (10)-(11),

$$U^o - U^b = [u(x^o) - x^o u'(x^o)] - [u(x) - xu'(x)] - \mu\left[u'(x^o) - \rho u'(x)\right] > 0,$$

for a sufficiently low μ. Thus, the policy combination of $\eta R^m = 1$ leads to complete disintermediation $(\theta = 0)$, which implies that the lower bound, $\frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$, must be higher than or equal to one. Also, $\frac{u'(\bar{x})}{\eta\left[u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta\right]} > \frac{1}{\eta + \beta\gamma}$ for any $\eta \geq 1$ because, as η rises, $\eta + \beta\gamma$ increases more than $\eta\left[u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta\right]$. Therefore, the ELB on the nominal interest rate is always determined by (50).

Proof of Proposition C.2: First, note that an equilibrium with $\theta = 0$ exists only when R^m is set at the ELB defined in (16). As mentioned in the Proof of Proposition C.1, any R^m higher than the ELB implies $x^o < \bar{x}$, where $\bar{x}u'(\bar{x}) = v$. Since $x^o u'(x^o) < v$ and the collateral constraint must bind in an equilibrium where v is sufficiently low, there must exist some buyers participating in banking contracts, i.e., $\theta > 0$. Thus, for any R^m higher than the ELB, $\theta > 0$ in equilibrium.

Next, consider an equilibrium with $0 < \theta < 1$. In this case, (x^c, x^d, x^o, θ) must satisfy:

$$\eta R^m = \frac{u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \tag{52}$$

$$(1-\rho)\theta x^d \left[u'(x^d) + \frac{\delta}{1-\delta}\right] + \rho\theta x^c \left[u'(x^c) + \frac{\delta}{1-\delta}\right] + (1-\theta)x^o u'(x^o) = v, \tag{53}$$

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \tag{54}$$

$$\rho\left[u(x^c) - x^c u'(x^c)\right] + (1-\rho)\left[u(x^d) - x^d u'(x^d)\right] = u(x^o) - x^o u'(x^o). \tag{55}$$

Suppose that there is an increase in ηR^m. Then, from (52), x^c decreases while x^d increases. From (54), x^o decreases with the decrease in x^c and the increase in x^d. Also, from (16) and (54), a necessary condition for this equilibrium to exist is $x^c < x^d$, implying that U^b decreases as x^c falls and x^d rises. As the left-hand side of (55) (representing U^b) decreases, x^o must fall in equilibrium, consistent with (54). Then, from (53), θ must rise in equilibrium. The effects of an increase in ηR^m on $(\pi, \alpha^b, \alpha^s)$ can be derived from (19), (22), and (23).

Given that an increase in ηR^m leads to a decrease in x^c and x^o and an increase in x^d and θ, there exists a ηR^m, denoted by $\bar{\Omega}$, that satisfies equation (52), with x^c and x^d that consist of the solution to equations (53)-(55) when $\theta = 1$. Therefore, I can conclude that, in equilibrium, $0 \le \theta < 1$ if $\frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta} \le \eta R^m < \bar{\Omega}$ and $\theta = 1$ if $\eta R^m \ge \bar{\Omega}$, where $\bar{x}$ and $\underline{x}^d$ are the quantities defined in the Proof of Proposition A.3.1.

Proof of Proposition C.3: In the Proof of Proposition C.2, I have shown that, for any $\eta R^m > \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$, the fraction θ is positive and $x^c < x^d$ in equilibrium. This implies that both the left-hand and the right-hand side of (55) increase (i.e., both U^b and U^o increase) as x^c and x^o rise and x^d falls. So, given η, a decrease in R^m increases the welfare measure $\mathcal{W}$ by increasing x^c and x^o and decreasing x^d and θ. Then, by continuity, the maximum $\mathcal{W}$ can be obtained when $\eta R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d)-\delta u'(\underline{x}^d)+\delta}$ given η. However, if the central bank conducts monetary policy (R^m, η) such that $\eta R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d)-\delta u'(\underline{x}^d)+\delta}$, a higher η only implies a higher α^b without increasing the sum of surpluses from trade in the CM and the DM. As a higher α^b leads to a larger total cost of theft, the welfare measure $\mathcal{W}$ is maximized if and only if $\eta = 1$ and $R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d)-\delta u'(\underline{x}^d)+\delta}$.

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