Negative Nominal Interest Rates and Monetary Policy*

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Abstract

How much can central banks reduce nominal interest rates? Can the lower bound be controlled by monetary policy? If so, should central banks reduce it to implement negative interest rates? I construct a model with multiple means of payment where the costs of holding paper currency effectively reduce its rate of return, creating a negative effective lower bound on interest rates. I find that central banks can reduce this lower bound with *a non-par exchange rate* between currency and bank reserves, but doing so raises currency-holding costs for individuals, leading to welfare losses. Moreover, implementing a negative rate by reducing the lower bound has no benefits because this policy combination lowers both the rate of return on currency and the interest rate on financial assets, leaving relative interest rates unchanged.

Keywords: Negative interest rate; Effective lower bound; Money; Banking; Monetary policy

JEL Codes: E4; E5

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1 Introduction

Since the global financial crisis, negative nominal interest rates have become a part of the monetary policy toolkit for some prominent central banks in the world such as the European Central Bank, the Swiss National Bank, the Swedish Riksbank, the Bank of Japan, and the National Bank of Denmark. Although negative nominal interest rates are feasible, the lower bound on nominal interest rates can still be an important binding constraint on monetary policy.[1] Accordingly, there has been much discussion about policy tools aimed at reducing this lower bound. Examples include imposing a limit on cash withdrawals, eliminating large-denomination bills, and introducing a non-par exchange rate between currency and reserves (Goodfriend, 2016; Rogoff, 2017a,b; Agarwal and Kimball, 2015, 2019).

Understanding how and why central banks could control the lower bound requires uncovering its determinants. What economic fundamentals or frictions determine the lower bound? Would it be desirable to manipulate these frictions to reduce the lower bound? I answer these questions by developing a model where the lower bound on nominal interest rates is endogenously negative and responds to central bank policies. My main finding is that central banks can reduce this lower bound, but such efforts only result in welfare losses.

In standard macroeconomic theory, monetary policy is constrained by the zero lower bound on nominal interest rates.[2] This constraint arises because of arbitrage: borrowing at a negative rate and investing in zero-interest paper currency would be profitable if nominal interest rates were negative. However, as noted above, negative nominal short-term interest rates are implementable in practice. This suggests the presence of frictions that inhibit arbitrage, making the lower bound on nominal interest rates effectively negative.

[1]The effective lower bound was a binding constraint for some central banks during the 2010s, as they have been cautious about implementing negative or substantially negative interest rates. Although the Swiss National Bank and the National Bank of Denmark successfully set a record-low policy rate of -0.75 percent, it remains uncertain whether they could have further reduced their policy rates and whether other central banks could set their policy rates at that level. For instance, Witmer and Yang (2016) find that the effective lower bound for the Bank of Canada policy rate ranges between -0.25 percent and -0.75 percent.

[2]See, for example, Woodford (2003) and Curdia and Woodford (2010) for standard New Keynesian models. See, also, Lagos and Wright (2005) and Lagos, Rocheteau, and Wright (2017) for standard New Monetarist models.

The frictions that create a negative effective lower bound are inherent to paper currency and emerge from the costs associated with storing, transporting, and exchanging it in large quantities. For example, if a private bank faces a negative nominal interest rate on reserves, it might contemplate holding currency instead of having reserve balances with the central bank. However, this would entail costs such as installing a sizable vault and hiring security guards to watch it. Moreover, the currency held would be of little or no use in making interbank and online transactions.

The policy tools aimed at reducing the effective lower bound essentially work by enhancing or taking advantage of these frictions to reduce the effective rate of return on holding currency.[3] A lower return on currency leads to a lower effective lower bound, which allows potentially welfare-enhancing monetary policy. However, reducing the currency's effective rate of return also implies an increased cost of using currency as a means of payment and a potential welfare loss. I formally examine this previously unexplored tradeoff to evaluate the welfare implications of reducing the effective lower bound on nominal interest rates.

Specifically, I develop a model with multiple assets—currency, bank deposits, reserves, and government bonds—and introduce currency-holding costs that lead to a negative effective lower bound. Also, to demonstrate how a central bank can reduce this effective lower bound, I consider a market-based reserve policy. This policy involves altering the one-to-one exchange rate between currency and bank reserves, as proposed by Eisler (1932), Buiter (2010) and Agarwal and Kimball (2015).[4] As the central bank can set a different exchange rate for private banks' current currency withdrawals from the one for their future currency deposits, it can reduce the nominal rate of return on currency faced by private banks.

[3]Some policy tools directly affect the currency-holding costs mentioned earlier. Imposing a quantitative limit on cash withdrawals from the central bank cash window or eliminating large-denomination bills increases the storage and/or transportation costs of currency. Other policy tools, including a non-par exchange rate between currency and reserves that I study in this paper, are intended to indirectly increase the cost of holding currency by reducing its nominal rate of return.

[4]In particular, Agarwal and Kimball (2019) consider this unconventional reserve policy as "the first-best approach with the fewest undesirable side-effects" because the nominal rate of return on currency in units of reserves, created by the central bank, can be naturally transmitted to the rate of return on currency in units of other financial assets.

For this reserve policy to be effective, it must alter the nominal rate of return on currency not only for private banks but also for private individuals. I show that this policy—a *non-par* exchange rate for currency withdrawals—becomes effective when currency-holding costs incentivize individuals to continue depositing and withdrawing currency, even though this activity yields a negative nominal return. Given the presence of currency-holding costs, the central bank can reduce the rate of return on currency for both private banks and individuals, and therefore, it can successfully lower the effective lower bound.

The ability of a non-par exchange rate to reduce the effective lower bound, however, can be limited. If individuals choose not to deposit their currency at banks, the central bank loses its ability to control the effective lower bound. This can happen because a non-par exchange rate incentivizes currency side-trading in the private sector. Specifically, a non-par exchange rate encourages private banks to acquire currency from the private sector rather than withdrawing it from the central bank. Since private banks are willing to acquire currency at a price (in units of reserves) lower than the non-par exchange rate, and private individuals may be willing to sell their currency at a price higher than one unit of reserves, currency side-trading can emerge endogenously. However, because engaging in side trades incurs currency-holding costs for individuals, they do so only when the marginal benefit from this activity is sufficiently high—that is, when the non-par exchange rate for currency withdrawals is sufficiently high.

This result can be reversed if the marginal cost of currency side-trading increases with currency holdings. This can happen endogenously, for instance, when a larger amount of currency held by individuals promotes currency theft. That is, the cost of holding currency can increase at the margin either because theft occurs more frequently or because individuals must incur higher security costs to prevent it. I model this by allowing individuals to steal currency, at a cost, from those engaged in side-trades, so that theft arises endogenously.[5] The resulting increase in the cost of holding currency encourages individuals to deposit their

[5] An endogenous cost of holding currency in the form of theft risk is also incorporated in He, Huang, and Wright (2005, 2008) and Sanches and Williamson (2010), but in a different manner.

currency at a one-to-one exchange rate. Consequently, the central bank can always reduce the effective lower bound on nominal interest rates.

I then turn to the consequences of reducing the effective lower bound and implementing a lower nominal interest rate on reserves. Currency-holding costs create inefficiencies in ordinary transactions by reducing the quantity of currency used as a means of payment, which decreases production and consumption. In this environment, lowering the nominal interest rate can improve welfare as it effectively increases the real return on currency, thereby mitigating the inefficiencies in ordinary transactions.[6] However, reducing the effective lower bound encourages more individuals to engage in socially useless and costly currency side-trading and theft, decreasing welfare.

Despite this apparent tradeoff, I find that it is never optimal to reduce the effective lower bound, as doing so only increases the cost of holding currency without yielding any welfare gains. Intuitively, lowering the nominal interest rate on reserves encourages private banks to hold more currency relative to interest-bearing assets. However, reducing the effective lower bound, which involves lowering the nominal rate of return on currency, does the opposite: it incentivizes private banks to hold less currency relative to interest-bearing assets. So, if the central bank reduces both the interest rate and the lower bound by the same magnitude, the effect of reducing the lower bound completely offsets that of reducing the interest rate. In other words, *it is the interest rate on reserves relative to currency that determines the monetary policy stance.* Because the central bank cannot further reduce this relative interest rate, there are no gains from reducing the effective lower bound.

Finally, I explore how disintermediation can affect the effective lower bound. Disintermediation happens when a sufficiently low nominal interest rate makes more individuals decide not to hold banking contracts. This is a practical concern because bank deposits serve as a

[6]This is a standard property in monetary theory with implications for optimal monetary policy. Similar to He, Huang, and Wright (2008), I show that the presence of currency-holding costs can make the optimal nominal interest rate negative. See Lagos, Rocheteau, and Wright (2017) for more discussion about optimal monetary policy.

primary and stable funding source for financing bank loans, and thus, disintermediation could lead to long-run inefficiency in the financial system. To study the role of disintermediation, I extend my model by allowing individuals to opt out of the banking system and use only currency in ordinary transactions. I find that a non-par exchange rate policy enables central banks to set a negative nominal interest rate without causing disintermediation. However, this policy eventually results in welfare losses, as in the baseline model.

Literature Review—This paper contributes to the literature on how to reduce the effective lower bound on nominal interest rates. Agarwal and Kimball (2019) provide a comprehensive survey and discuss policy tools suggested in the literature. The idea of a non-par exchange rate between currency and reserves was first proposed by Eisler (1932) in the form of a dual currency system where one currency (physical currency) is used as a means of payment, and the other (electronic money) plays a unit-of-account role. Buiter (2010) revived Eisler's proposal with a simple model illustrating how the central bank could frictionlessly reduce the lower bound by adjusting the exchange rate between the two currencies. More recently, Agarwal and Kimball (2015), Goodfriend (2016), and Rogoff (2017a,b) have favorably discussed a non-par exchange rate as a potential policy tool.

Compared to previous studies, I develop a formal model that incorporates the determinants of the effective lower bound—an essential but previously unexplored aspect for understanding how to adjust it. The model also shows how the rate of return on currency faced by private banks can be transmitted to private individuals, providing new insights. Lastly, while a non-par exchange rate policy is often considered neutral aside from its impact on the effective lower bound, I demonstrate that its real effects can offset the benefits of reducing the nominal interest rate and even lead to welfare losses by encouraging socially undesirable currency side-trading.

This paper also relates to theoretical studies exploring the implications of negative nominal interest rates, including He, Huang, and Wright (2008), Rognlie (2016), Rocheteau, Wright, and Xiao (2018), Jung (2019), Ulate (2021), Eggertsson, Juelsrud, Summers, and

Wold (2022), Berentsen, van Buggenum, and Ruprecht (2023), and Abadi, Brunnermeier, and Koby (2023). He, Huang, and Wright (2008) develop a model of currency, banking, and theft in which individuals choose either currency or bank deposits as a means of payment, with currency being less safe due to theft risk. They show that a negative nominal interest rate can be optimal under certain conditions. Another related study is Eggertsson, Juelsrud, Summers, and Wold (2022), which shows that the lower bound on disintermediation-free interest rates (deposit rates) can be higher than the lower bound on arbitrage-free policy rates (short-term interest rates). They find that when the former lower bound is binding while the latter is not, implementing a negative nominal interest rate policy can have contractionary effects due to a breakdown in the pass-through of the policy rate.[7]

While these studies provide valuable insights into how negative nominal interest rate policies work, there has been no formal analysis of how such policies can be implemented and their economic implications. This is a critical issue given the scope of practically implementable negative interest rates. My paper addresses this gap in the literature by developing a model that explicitly incorporates the frictions that determine the lower bound on nominal interest rates, offering a comprehensive analysis of their implementation and effects.

2 Model

The basic structure of the model builds on Lagos and Wright (2005) and Rocheteau and Wright (2005), incorporating a banking sector following Williamson (2016, 2022). Time is indexed by $t = 0, 1, 2, ...$, and each period consists of three sequential sub-periods: the theft stage (TS), the centralized market (CM), and the decentralized market (DM). I introduce the TS into the standard framework to capture frictions that limit the size of currency

[7] Using the extended model, I also find that the lower bound on nominal interest rates preventing complete disintermediation can exceed the one preventing arbitrage. While my analysis abstracts from the conflict between these two distinct lower bounds—a driving force for the imperfect pass-through of negative policy rates in Eggertsson, Juelsrud, Summers, and Wold (2022)—the results suggest that a non-par exchange rate policy can reduce both lower bounds by increasing the cost of holding currency.

side-trades, as I will show later.

Private Agents—There are three types of private agents in the economy. First, there is a continuum of *buyers* with unit mass, each of whom has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[-H_t + u(x_t) - e_t\right], \tag{1}$$

where H_t is labor supply in the CM, x_t is consumption in the DM, e_t is labor effort in the TS, and $\beta \in (0,1)$ is the discount factor. The utility function $u(\cdot)$ is strictly increasing, strictly concave, and twice continuously differentiable, with $u'(0) = \infty$, $u'(\infty) = 0$, and $-\frac{xu''(x)}{u'(x)} < 1$.[8] There is also a continuum of *sellers* with unit mass, each with preferences

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(X_t^s - h_t^s\right), \tag{2}$$

where X_t^s is consumption in the CM and h_t^s is labor supply in the DM.

In the CM, sellers wish to consume but cannot produce, while in the DM, buyers wish to consume but cannot produce. In both the CM and the DM, one unit of labor supply is converted into one unit of perishable consumption good. This preference structure, combined with the frictions introduced later, creates a role for a medium of exchange: buyers acquire assets in the CM that serve as a medium of exchange in the DM. Conversely, in the TS, neither production nor consumption takes place, but buyers can invest in theft technology at the cost of labor effort.

Finally, there is a continuum of *private banks* who are active only in the CM. Each bank's preferences are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(X_t^b - H_t^b\right), \tag{3}$$

[8]The assumption on relative risk aversion ensures that asset demand is strictly increasing in its rate of return, i.e., a change in an asset's return always leads to a larger substitution effect than the income effect. This assumption also provides analytical tractability, as in its absence, the effects of monetary policies are generally ambiguous (Andolfatto and Williamson, 2015). Due to its economic interpretation and analytical convenience, this assumption has been widely adopted in the literature (e.g., Williamson, 2016, 2022; Kang, 2017; Wang, 2023; Altermatt and Wang, 2024).

where X_t^b and H_t^b represent consumption and labor supply, respectively. Banks can produce goods in the CM using the same linear production technology available to buyers.[9]

Fiscal Authority and Central Bank—The government consists of a fiscal authority and a central bank. The fiscal authority issues nominal government bonds, while the central bank issues currency and reserves by exchanging them for government bonds. Nominal government bonds, issued in the CM of period t, are one-period bonds that pay off at a gross rate of R_{t+1}^b. Currency is perfectly divisible, portable, and storable, with a nominal interest rate of zero. Reserves are account balances with the central bank, and one unit of reserves acquired in the CM of period t pays off R_{t+1}^m units of reserves at the beginning of the next CM. Private banks can withdraw currency from their reserve accounts at the central bank.

In addition to its standard policy tools, the central bank can set a *non-par* exchange rate between currency and reserves to generate a negative nominal return on currency, following the ideas presented in Eisler (1932), Buiter (2010) and Agarwal and Kimball (2015). Specifically, depositing one unit of currency with the central bank yields one unit of reserves, whereas withdrawing one unit of currency in period t requires $\eta_t \geq 1$ units of reserves.

Timing of Events—At the beginning of the CM, all debts are settled, and private banks deposit any remaining currency with the central bank. Then, all agents participate in a perfectly competitive market where production, consumption, and asset exchanges take place. Private banks can issue deposit liabilities and acquire asset portfolios, including currency. They can obtain currency through two channels: purchasing it in the competitive market at the prevailing market price or acquiring reserves and exchanging them for currency at the central bank cash window. Following Williamson (2016, 2022), all buyers are identical at this stage but face a publicly observable, idiosyncratic risk that determines the type of means of payment they will use in the subsequent DM. Once the CM closes, buyers' types

[9] It is common in the literature to model banks as agents with preferences. In early models such as Diamond (1984) and Williamson (1986), some agents (lenders) endogenously become banks because they can monitor borrowers more efficiently. In Cavalcanti and Wallace (1999) and Gu, Mattesini, Monnet, and Wright (2013), agents less constrained by limited commitment emerge as banks that issue liabilities (inside money) to facilitate trade.

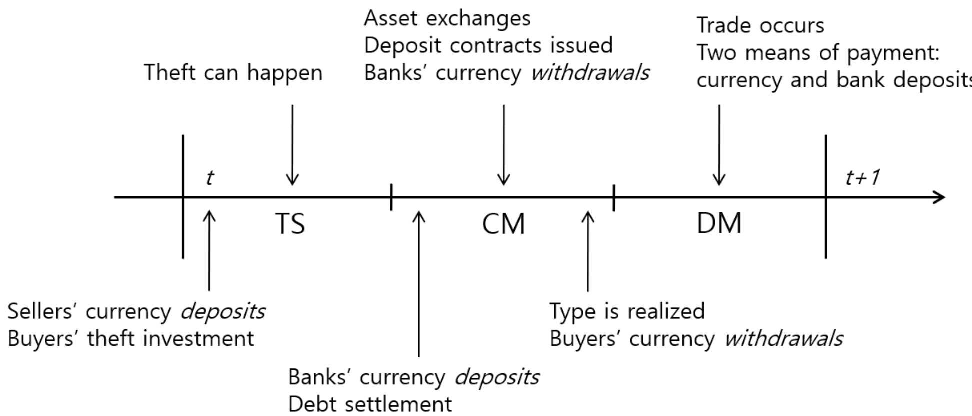

Figure 1: Timing of events

are realized, and each buyer can contact one bank. This idiosyncratic risk, combined with spatial separation, creates a role for banking arrangements to provide liquidity, similar to what happens in Diamond and Dybvig (1983).

In the DM, each buyer is randomly matched with a seller, and the terms of trade are determined by the buyer's take-it-or-leave-it offer. In any DM meeting, the matched buyer and seller do not know each others' histories—no memory or record-keeping—and no one can be forced to work due to limited commitment. As a result, unsecured credit is not incentive-compatible, as debtors would have no incentive to honor their obligations. Given this environment, each buyer needs a means of payment to purchase goods in the DM, but the type of payment depends on the buyer's type, which is realized after the CM. Specifically, with probability ρ, buyers must use currency as a means of payment, while with probability $1-\rho$, they can use any non-currency assets.[10] As will be discussed later, buyers choose to acquire banking contracts that allow them to either withdraw currency or hold deposit claims that can be accepted as a means of payment in the DM.[11]

[10]One interpretation is that some consumers value privacy, and using paper currency helps preserve the privacy of their transactions. Alternatively, it can be viewed as a market segmentation, where a fraction ρ of sellers accept only currency, while the remaining fraction $1-\rho$ accept non-currency assets.

[11]To make the model complete, assume that private banks have collateral technology that allows creditors to seize at least part of the bank assets in case of default.

At the beginning of the TS, sellers have the option to deposit their currency with the central bank in exchange for reserve balances.[12] If a seller deposits currency, one unit of currency is exchanged for one unit of reserves, which sellers can then use to purchase goods in the following CM. After sellers complete currency deposits, buyers have the opportunity to acquire theft technology (e.g., producing a weapon) by exerting ε units of labor effort. Subsequently, buyers and sellers are randomly matched in bilateral meetings. If a buyer equipped with theft technology encounters a seller carrying currency, the buyer can steal all of the seller's currency. The timing of events is summarized in Figure 1.

Currency Flows—The model is designed to generate currency flows within the economy and capture how a non-par exchange rate policy can distort these flows. In the absence of a non-par exchange rate policy, currency moves from the central bank to private banks in the CM, as banks demand currency to meet depositors' withdrawal requests. After the CM, currency flows from private banks to some buyers and, in the DM, from buyers to sellers. Then in the TS, sellers deposit their currency, ensuring that all old currency returns to the central bank before new currency is issued.

Introducing a non-par exchange rate, however, raises the cost of currency withdrawals, discouraging private banks from using the central bank cash window. Instead, banks may seek to acquire currency in a private market at a price lower than the non-par exchange rate. In turn, individuals holding currency may prefer selling it in the private market rather than depositing it at the one-to-one exchange rate. As a result, currency side trades emerge, preventing some old currency from returning to the central bank. This framework shows how a non-par exchange rate policy distorts banks' behavior and currency flows.

Costs of Holding Currency—The model also captures potential social costs associated with these side trades. Currency side trades may be socially undesirable due to the resource

[12] In practice, individuals cannot hold reserve accounts directly with the central bank. One interpretation of this assumption is that individuals deposit currency with a private bank, which then exchanges it for reserve balances. An alternative interpretation is that individuals deposit currency with the central bank, which credits the corresponding private bank, and the bank, in turn, credits each individual's bank account.

costs required to secure these transactions. The TS specifically accounts for the costs of trading currency in the form of theft risk: sellers who do not deposit their currency to engage in side trades face the risk of losing it in the process.[13]

In addition to the risk of theft, I introduce two other forms of currency-holding costs in the CM and the DM to illustrate their implications for the lower bound on nominal interest rates and optimal monetary policy. First, any agent can hold currency across periods, but doing so incurs a proportional cost γ in the CM. Specifically, holding C_t units of currency from the CM of period t to the next CM entails a cost of γC_t units in period $t+1$, with $\gamma \in (0, 1)$. This storage cost has implications for the lower bound on nominal interest rates because it effectively lowers the rate of return on currency. Second, individuals holding currency at the end of the DM face a fixed cost μ. This cost introduces inefficiencies in DM transactions, which alters the central bank's optimal interest rate policy.[14]

Importantly, the marginal costs of holding currency in the CM and the DM are constant, whereas the marginal cost associated with theft risk in the TS is determined endogenously. As I will demonstrate later, the effectiveness of a non-par exchange rate policy crucially depends on the extent to which this marginal cost responds to the policy.

Government's Budget Constraint—After the debt settlement stage in each CM, the fiscal authority issues nominal government bonds and makes transfers to each buyer. At the same time, the central bank transfers any profits or losses to the fiscal autority and issues reserves by exchanging them for government bonds. Private banks can withdraw currency from their reserve accounts by exchanging $\eta_0 \geq 1$ units of reserves per unit of currency. Let $\bar{C}_t$, $\bar{M}_t$, and $\bar{B}_t$ denote the quantities of currency, reserves, and nominal government bonds outstanding at the end of the CM of period t, and let τ_t and ϕ_t denote the quantity

[13]The timing of events is carefully designed to capture the potential social costs arising from currency side trades. If the TS occurs after the CM, private banks will bear the security costs of holding currency in equilibrium regardless of how the currency is acquired. Thus, the model will fail to capture the social costs associated with currency side trades.

[14]Introducing a fixed cost is sufficient to generate inefficiencies in currency-involved DM transactions. Adding a proportional cost in the DM would make the model analytically intractable.

of transfers and the price of money in units of goods, respectively. Then, the consolidated government's budget constraints are given by

$$\phi_0 \left(\eta_0 \bar{C}_0 + \bar{M}_0 + \bar{B}_0\right) = \tau_0, \qquad \text{for } t = 0 \tag{4}$$

$$\phi_t \left(\eta_t \bar{C}_t + \bar{M}_t + \bar{B}_t\right) = \phi_t \left(\bar{C}_{t-1} + R_t^m \bar{M}_{t-1} + R_t^b \bar{B}_{t-1}\right) + \tau_t. \qquad \text{for } t \geq 1 \tag{5}$$

The left-hand sides of the above equations represent the government's total revenue from issuing new liabilities, while the right-hand sides represent total expenditure, including debt repayments and transfers to buyers (or taxes if $\tau_t < 0$). Note that currency issued in period $t-1$ and deposited in period t is converted into reserves at a one-to-one exchange rate, whereas newly issued currency is exchanged for reserves at the non-par rate η_t.

In this model, reserves serve as the unit of account. This setup resembles Eisler (1932)'s dual currency system, where physical money functions as a medium of exchange while electronic money serves as the unit of account. However, unlike Eisler's framework, electronic money—such as deposit claims—also acts as a medium of exchange in my model.

Government's Policies—In addition to the non-par exchange rate policy described earlier, the central bank conducts monetary policy under a floor system. Under this system, the central bank administratively sets the nominal interest rate on reserves R_t^m, which influences the nominal interest rate on bonds R_t^b, provided that private banks hold a sufficiently large quantity of reserves. Also, the central bank can expand its balance sheet through swaps of reserves for government bonds. Let $\omega_t = \phi_t \left(\eta_t \bar{C}_t + \bar{M}_t\right)$ denote the size of the central bank's balance sheet, in real terms.[15]

As in Andolfatto and Williamson (2015) and Williamson (2016), the fiscal authority

[15]The real value of the central bank's assets equals that of its liabilities because the central bank is assumed to transfer any profits or losses to the fiscal authority, maintaining a net worth of zero.

determines the real value of the consolidated government debt outstanding, denoted as v_t,

$$v_t = \phi_t \left(\eta_t \bar{C}_t + \bar{M}_t + \bar{B}_t \right), \tag{6}$$

for all $t \geq 0$. The value v_t can be interpreted as a debt ceiling because, in this model, a benevolent social planner constrained by $\tilde{v}_t \leq v_t$ would always choose $\tilde{v}_t = v_t$ to maximize social welfare.[16] To achieve its policy objective, the fiscal authority adjusts lump-sum transfers τ_t in equations (4) and (5). This fiscal policy rule is a useful specification for generating low real interest rates in equilibrium. As will be shown later, a low level of consolidated government debt (or a low v_t) implies a limited supply of assets that can be used as collateral to back banks' liabilities. This leads to binding collateral constraints, a liquidity premium on those assets, and consequently, low real interest rates.

3 Equilibrium

I focus on stationary equilibria where all real variables and government policies are constant across periods, and the inflation rate is $\pi = \phi_t / \phi_{t+1}$ for all $t \geq 0$. I first describe agents' optimal behavior to formally define an equilibrium. Then, I characterize equilibrium conditions and demonstrate why the effective lower bound on nominal interest rates can be negative.

3.1 Optimization

In this section, I describe individuals' optimal decisions in the TS and DM, as well as private banks' optimal banking arrangements in the CM.

[16] The value v_t is equivalent to the total quantity of bonds, in real terms, issued by the fiscal authority. In period 0, the central bank purchases bonds by issuing currency and reserves, and in every subsequent period, it transfers its profits to the fiscal authority. This implies that the central bank's total assets must always equal its total liabilities, that is, $\phi_t(\eta_t \bar{C}_t + \bar{M}_t) = \phi_t \hat{B}_t$, where $\hat{B}_t$ denotes the quantity of bonds held by the central bank. Substituting this into equation (6) yields $v_t = \phi_t(\hat{B}_t + \bar{B}_t)$, which is the total quantity of bonds issued by the fiscal authority.

3.1.1 Currency Side-Trading and Theft

Private banks can withdraw currency from the central bank by paying η units of reserves per unit of currency. However, if currency holders in the CM are willing to sell their currency for less than η units of reserves, private banks will prefer to purchase it from them. In the CM, the price of currency will naturally be determined by its supply and demand. Here, I focus on cases where there is inflation in equilibrium, i.e., $\pi > 1$. With inflation, the real quantity of currency supplied in the CM consistently falls short of the quantity demanded. As a result, the price of currency in units of reserves will be η, causing banks to be indifferent between withdrawing currency from the central bank and purchasing it from sellers.[17]

Let α^s denote the fraction of sellers who bring currency into the TS among those who obtained it in the previous DM. Currency side-trading in the CM occurs only if there exist sellers who decided not to deposit their currency in the previous TS, i.e., $\alpha^s > 0$. However, in the TS, some buyers may exert ε units of labor effort to invest in theft technology. Let α^b denote the fraction of such buyers. Suppose that the representative currency-holding seller carries c^s units of currency (in real terms) into the TS. Since theft technology requires costly investment, each buyer's decision must be incentive-compatible in equilibrium:

$$\alpha^b = \begin{cases} 0 & \text{if } \varepsilon > \rho\alpha^s\eta c^s \\ \in [0,1] & \text{if } \varepsilon = \rho\alpha^s\eta c^s \\ 1 & \text{if } \varepsilon < \rho\alpha^s\eta c^s \end{cases} \tag{7}$$

Here, $\rho\alpha^s$ is the probability of meeting a currency-holding seller, while ηc^s is the CM value of currency held by the seller. These conditions imply that theft does not occur if buyers strictly prefer not to invest in theft technology, occurs sometimes if they are indifferent, and always occurs if theft is strictly preferred.

[17] Specifically, I focus on stationary equilibria with sufficiently low values of v and γ, so that $\pi > 1$ in equilibrium. See Appendix A.2 for the conditions that lead to stationary equilibria with inflation.

Similarly, each seller's decision in the TS must also be incentive-compatible:

$$\alpha^s = \begin{cases} 0 & \text{if } (1-\alpha^b)\eta < 1 \\ \in [0,1] & \text{if } (1-\alpha^b)\eta = 1 \\ 1 & \text{if } (1-\alpha^b)\eta > 1 \end{cases} \tag{8}$$

Sellers never bring currency into the TS if its expected per-currency payoff, $(1-\alpha^b)\eta$, is lower than the guaranteed one-unit reserve payoff from making deposits. If the payoffs are identical, sellers sometimes carry currency. If carrying currency is more profitable, sellers always do it.

3.1.2 Deposit Contracts

Buyers could acquire currency and government bonds in the CM and then use either as a means of payment in the following DM. However, private banks endogenously design deposit contracts that efficiently allocate liquid assets across different DM meetings, as in Williamson (2016, 2022). These contracts, written in the CM, provide insurance to buyers by giving them the option to withdraw currency after the CM, once they learn their types. Buyers in need of currency visit banks to make withdrawals, while those who do not exercise this option use deposit claims as a means of payment in the DM.

Each bank offers a deposit contract (k, c', d), where k is the quantity of goods deposited by each buyer in the CM, c' is the real quantity of currency that the buyer can withdraw after the CM, and d is the quantity of claims to goods in the following CM if no currency is withdrawn. The bank then acquires an asset portfolio (b, m, c), where b is the quantity of government bonds, m is the quantity of reserves, and c is the quantity of currency, all in real terms. Like all individuals in the economy, banks are subject to limited commitment, implying that they must pledge their assets as collateral to secure deposit liabilities.

Each bank chooses its deposit contract and asset portfolio to maximize the payoff from these banking activities. However, its contract must be at least as attractive as other contracts available in the CM to be chosen by buyers. Due to perfect competition, the bank's optimization problem in equilibrium is equivalent to the following:

$$\max_{k,c',d,b,m,c} \left\{-k + \rho u\left(x^c\right) + (1-\rho) u\left(x^d\right)\right\} \tag{9}$$

subject to

$$k - b - m - \eta c + \beta\left[-(1-\rho)d + \frac{R^m m + R^b b + c - \rho c'}{\pi} - \gamma(c - \rho c')\right] = 0, \tag{10}$$

$$-(1-\rho)d + \frac{R^m m + R^b b + c - \rho c'}{\pi} \geq \frac{\delta\left(R^m m + R^b b + c\right)}{\pi}, \tag{11}$$

$$k, c', d, b, m, c, c - \rho c' \geq 0. \tag{12}$$

The bank's objective function (9) is the representative buyer's expected utility from obtaining the deposit contract. With probability ρ, the buyer learns that they need currency and withdraws c' units from the bank. In the subsequent DM, the buyer is matched with a seller and trades the currency for x^c units of goods. With probability $1-\rho$, currency is not needed for transactions, so the buyer instead receives a claim to d units of goods. Then in the DM, the buyer exchanges these claims for x^d units of goods. As I will show later, the consumption quantities, x^c and x^d, are determined by the buyer's offer.

Equation (10) ensures that the bank earns zero discounted net payoff from its banking activities each period. In the CM, the bank receives k units of deposits from each buyer and acquires an asset portfolio of government bonds b, reserves m, and currency c. After the CM, it distributes currency to the fraction ρ of buyers who withdraw c' units each. The remaining fraction $1-\rho$ of buyers receive deposit claims, which the bank redeems for d units of goods in the next CM. The bank retains $c - \rho c'$ units of currency until the next CM, incurring $\gamma(c - \rho c')$ units of a storage cost. At the beginning of the next CM, it must deposit the currency with the central bank at a one-to-one exchange rate. Since any contract yielding a positive discounted net payoff cannot be sustained in equilibrium, the bank's payoff from these banking activities becomes zero.

Inequality (11) represents the collateral constraint faced by the bank. If the bank chooses to default, it refuses to deliver currency after the CM and absconds with a fraction δ of its asset holdings in the next CM. To ensure repayment, the bank must weakly prefer honoring its deposit liabilities in both after the CM and in the next CM (the left-hand side of 11) over defaulting and absconding with assets (the right-hand side). Finally, constraint (12) states that all real quantities must be nonnegative.

3.1.3 Terms of Trade

In the DM, each buyer makes a take-it-or-leave-it offer, which allows them to extract all surplus from trade. This means that, given the quantity of currency or deposit claims offered to the seller, the buyer can increase the quantity of goods produced by the seller until the seller receives zero surplus from trade.

If the seller receives c' units of currency (in real terms) from the buyer, it incurs a fixed cost μ at the end of the DM. At the beginning of the next TS, the seller holds $\frac{c'}{\pi}$ units of currency in terms of the next CM good. Then, with probablity $1-\alpha^s$, the seller deposits the currency and receives one unit of reserves per unit of currency. However, with probability $\alpha^s(1-\alpha^b)$, the seller successfully carries the currency into the next CM and sells it at price η. Therefore, the ex-ante expected payoff for the seller from receiving one unit of currency is $1-\alpha^s+\alpha^s(1-\alpha^b)\eta$.

Since the buyer's offer leaves the seller with zero surplus from trade, the quantity of goods produced by the seller in these transactions, x^c (or equivalently, the seller's disutility from production), equals the seller's discounted expected net payoff from acquiring c' units of currency. Conversely, in DM meetings where deposit claims are used as a means of payment, the quantity of goods produced by the seller, x^d, equals the seller's discounted net payoff from receiving d units of deposit claims:

$$x^c = [1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta]\frac{\beta c'}{\pi} - \mu. \tag{13}$$

$$x^d = \beta d. \tag{14}$$

3.2 Definition of Equilibrium

In a stationary equilibrium, the consolidated government's budget constraints (4 and 5) are

$$\eta\bar{c} + \bar{m} + \bar{b} = \tau_0, \tag{15}$$

$$\eta\bar{c} + \bar{m} + \bar{b} = \frac{\bar{c} + R^m\bar{m} + R^b\bar{b}}{\pi} + \tau, \tag{16}$$

where $\bar{c}$, $\bar{m}$, and $\bar{b}$ denote the real quantities of currency, reserves, and nominal government bonds outstanding at the end of each CM. Then, the fiscal policy rule (6) can be written as:

$$v = \eta\bar{c} + \bar{m} + \bar{b}. \tag{17}$$

In equilibrium, private agents optimize their decisions given asset prices. Specifically, individuals' behavior in the TS must be incentive-compatible (7 and 8). In the CM, banks optimize their decisions by solving their contracting problem (9). Buyers choose the terms of trade to maximize their DM trade surplus (13 and 14).

All asset markets must clear, meaning that the demand for each asset equals its supply:

$$c = \bar{c}; \qquad m = \bar{m}; \qquad b = \bar{b}. \tag{18}$$

Finally, there must be no arbitrage opportunities from holding currency from the CM to the next CM. From the first-order conditions of the bank's problem, I obtain the following no-arbitrage condition for banks:

$$-\eta+\beta\left(\frac{1}{\pi}-\gamma\right)+\frac{\lambda\left(1-\delta\right)}{\pi}\leq 0, \tag{19}$$

where λ is the Lagrange multiplier associated with the collateral constraint. The cost of acquiring an additional unit of currency in the CM is its price η. In the next CM, this currency incurs a proportional cost γ, and its real value declines to $1/\pi$ due to inflation. Since banks pledge $1-\delta$ of currency as collateral, the last term captures its collateral value.

Similarly, the expected payoffs for sellers and buyers from storing currency across periods must be non-positive at the margin. These no-arbitrage conditions are

$$-\eta+\beta\left[\frac{1-\alpha^s+\alpha^s(1-\alpha^b)\eta}{\pi}-\gamma\right]\leq 0, \tag{20}$$

$$-\eta+\beta\left(\frac{1}{\pi}-\gamma\right)\leq 0. \tag{21}$$

Inequality (20) applies to sellers, while (21) applies to buyers. The returns to holding currency may differ for buyers and sellers. Consistent with their equilibrium behavior, sellers either deposit their currency with probability $1-\alpha^s$, receiving one unit reserves per unit of currency, or successfully carry it into the next CM with probability $\alpha^s(1-\alpha^b)$, where they sell it at price η. By contrast, buyers deposit their currency with the central bank at the one-to-one exchange rate.[18] Formally, an equilibrium is defined as follows.

Definition *Given fiscal policy v and monetary policy (R^m,ω,η), a stationary equilibrium consists of DM consumption quantities (x^c,x^d), CM asset quantities (k,c',d,b,m,c), the fraction of buyers who acquire theft technology in the TS α^b, the fraction of currency-holding*

[18]This assumption allows us to focus on sellers' behavior in the TS while keeping the model tractable. One could consider relaxing this assumption by allowing buyers to decide whether to deposit their currency at the beginning of the TS. However, this would require assuming that both buyers and sellers can invest in theft technology. Such a modification introduces various possibilities, including scenarios where a currency-holding seller equipped with theft technology meets a buyer with theft technology. This would only complicate the model without changing the main results of the paper.

sellers who carry it into the TS α^s, transfers (τ_0, τ), the inflation rate π, and the nominal interest rate on bonds R^b that solve each bank's contracting problem (9) and satisfy incentive compatibility conditions (7)-(8), each seller's participation constraints (13)-(14), the consolidated government's budget constraints (15)-(16), the fiscal policy rule (17), market-clearing conditions (18), and no-arbitrage conditions (19)-(21).

Fiscal and monetary policies are exogenous. The fiscal authority determines the total value of consolidated government debt, while the central bank targets three policy instruments: (i) the nominal interest rate on reserves R^m, (ii) the size of the central bank's balance sheet ω, and (iii) the non-par exchange rate for currency withdrawals η. In equilibrium, the nominal interest rate on reserves effectively determines the nominal interest rate on bonds, i.e., $R^b = R^m$. This arises because private banks treat reserves and government bonds as identical assets at the margin.[19] Given these monetary policy targets, the fiscal authority adjusts transfers τ_0 and τ to achieve its fiscal policy target v while ensuring that the consolidated government's budget constraints hold.

3.3 Effective Lower Bound on Nominal Interest Rates

In this model, the set of nominal interest rates, R^m, that can be sustained in equilibrium is determined endogenously. I define the effective lower bound (ELB) as the minimum sustainable interest rate that ensures private banks and individuals do not have arbitrage opportunities from holding currency across periods.

Using the first-order conditions of a bank's problem and inequalities (19)-(21), I obtain the set of nominal interest rates that prevent arbitrage activities for banks, sellers, and

[19] In practice, reserves serve as a useful means of payment in intraday trading within the banking system. However, since the global financial crisis, a large volume of reserves has been held by private banks without being used in intraday transactions. This justifies abstracting from the role of reserves as a means of payment.

buyers, respectively, as

$$R^m \geq \frac{1}{\eta + \beta\gamma}, \tag{22}$$

$$R^m \geq \frac{1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta}{(\eta + \beta\gamma)[(1 - \delta)u'(x^d) + \delta]}, \tag{23}$$

$$R^m \geq \frac{1}{(\eta + \beta\gamma)[(1 - \delta)u'(x^d) + \delta]}. \tag{24}$$

In equilibrium, R^m must be sufficiently high to satisfy all the above inequalities. Since any interest rate that prevents arbitrage for sellers also ensures no arbitrage for buyers, the ELB is defined as

$$R^m \geq \max\left\{\frac{1}{\eta + \beta\gamma}, \frac{1 - \alpha^s + \alpha^s(1 - \alpha^b)\eta}{(\eta + \beta\gamma)[(1 - \delta)u'(x^d) + \delta]}\right\} \equiv ELB. \tag{25}$$

To illustrate why the gross nominal interest rate R^m can be less than one (or equivalently, why the net nominal interest rate $R^m - 1$ can be negative), consider first the case where the ELB is determined by the first argument in (25). In this case, the ELB can fall below one for two reasons. First, the proportional cost associated with holding currency across periods ($\gamma > 0$) lowers the effective rate of return on currency, causing the ELB to fall below one. This result helps explain how some central banks have implemented negative nominal interest rates without triggering immediate arbitrage transactions or disruptions in the financial system. Second, a non-par exchange rate for currency withdrawals ($\eta > 1$) also pushes the ELB below one. When the exchange rate for currency withdrawals exceeds that for deposits, the nominal rate of return on holding currency across periods declines. This mechanism enables R^m to fall below one without inducing arbitrage.

Now, consider instead the case where the ELB is determined by the second argument. In this case, a non-standard mechanism can drive the ELB below one. As I will show later, the term $(1 - \delta)u'(x^d) + \delta$ is greater than one due to low real interest rates on collateralizable assets. Real interest rates are low because assets used as collateral carry a liquidity premium, driving up asset prices. However, sellers do not receive this non-monetary payoff from using currency as collateral, so the effective rate of return on currency perceived by sellers is lower than that perceived by banks. When real interest rates are sufficiently low, the ELB can be lower than one.

3.4 Characterization of Equilibrium

In equilibrium, sellers who receive currency in the DM hold c'/π units of currency at the beginning of the following TS. They decide whether to bring the currency into the TS based on the probability of meeting a thief, α^b, and the price of currency in the CM, η. Meanwhile, buyers decide whether to invest in theft technology given the probability of meeting a currency-carrying seller, $\rho\alpha^s$, and the price of currency, η. The following lemma characterizes the equilibrium behavior of individuals in the TS.

Lemma 1 (Existence of Theft) *1. Suppose $\eta = 1$. Then, no theft occurs, i.e., $\alpha^b = 0$. Meanwhile, $\alpha^s \in [0, 1]$ for $\varepsilon \geq \frac{\rho\eta c'}{\pi}$, and $\alpha^s \in [0, \bar{\alpha}^s]$ for $\varepsilon < \frac{\rho\eta c'}{\pi}$, where $\bar{\alpha}^s = \frac{\pi\varepsilon}{\rho\eta c'}$.*
2. Suppose $\eta > 1$. Then, i) for $\varepsilon \geq \frac{\rho\eta c'}{\pi}$, no theft occurs, with $\alpha^b = 0$ and $\alpha^s = 1$ and ii) for $\varepsilon < \frac{\rho\eta c'}{\pi}$, theft occurs, with $\alpha^b = \frac{\eta-1}{\eta}$ and $\alpha^s = \frac{\pi\varepsilon}{\rho\eta c'}$.

According to Lemma 1, theft does not occur in equilibrium when private banks can withdraw currency from their reserve accounts at par ($\eta = 1$). In this case, the price of currency is identical to that of reserves in terms of goods, making sellers indifferent between depositing currency with the central bank and engaging in side trades with private banks. As a result, sellers increase the probability of depositing their currency until theft becomes unprofitable, so as to eliminate theft in equilibrium.

Lemma 1 also shows that introducing a non-par exchange rate ($\eta > 1$) incentivizes sellers to engage in currency side-trading. If the effort cost of investing in theft technology ε is sufficiently high, buyers do not steal currency ($\alpha^b = 0$), making sellers strictly prefer to carry currency into the TS rather than depositing it with the central bank ($\alpha^s = 1$). Conversely, if

ε is sufficiently low, buyers have an incentive to steal currency, increasing the marginal cost for sellers carrying currency into the TS. In this case, equilibrium occurs when both buyers and sellers become indifferent between their available choices.

In this model, when v (the real value of the consolidated government debt outstanding) is sufficiently low, private banks cannot acquire a sufficient quantity of collateralizable assets to offer buyers the first-best quantity of deposit claims. In this case, private banks' collateral constraints bind, and the consumption quantity in deposit-based DM transactions x^d becomes inefficiently small. In what follows, I will focus on cases where v is sufficiently low so that the collateral constraint (11) binds in equilibrium.[20]

3.4.1 Equilibrium with No Theft

In this section, I analyze an equilibrium in which there is no threat of theft in the TS. This type of equilibrium arises when the cost of theft ε is sufficiently high to deter buyers from investing in theft technology.[21] Suppose ε is sufficiently high to ensure that $\alpha^b = 0$ and $\alpha^s = 1$ in equilibrium. Then, from the first-order conditions of the bank's problem and Lemma 1, the inflation rate π and the nominal interest rates on reserves R^m and government bonds R^b can be expressed as

$$\pi = \frac{\beta}{\eta}\left[\eta u'(x^c) - \delta u'(x^d) + \delta\right], \tag{26}$$

$$R = R^m = R^b = \frac{\eta u'(x^c) - \delta u'(x^d) + \delta}{\eta\left[u'(x^d) - \delta u'(x^d) + \delta\right]}, \tag{27}$$

while the corresponding real interest rates can be written as

$$r = r^m = r^b = \frac{1}{\beta\left[u'(x^d) - \delta u'(x^d) + \delta\right]}. \tag{28}$$

[20]Characterizing an equilibrium with a non-binding collateral constraint is straightforward. When constraint (11) does not bind, the associated Lagrange multiplier becomes zero, and it follows that $u'(x^d) = 1$. Readers who wish to analyze such an equilibrium may simply replace $u'(x^d)$ with one and disregard the equation related to the collateral constraint in the following sections.

[21]Studying such an equilibrium is analogous to analyzing an equilibrium in a model without a TS or in a model where the marginal cost of holding currency in the TS remains invariant to policies.

The consumption quantity in deposit-based DM meetings x^d is inefficiently low due to binding collateral constraints (represented by $u'(x^d) > 1$), leading to low real interest rates.

Using equations (13)-(18) and the first-order conditions, I can rewrite the collateral constraint (11) as

$$\underbrace{\left[u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\rho(x^c+\mu)}_{\text{Demand for currency}} + \underbrace{\left[u'\left(x^d\right) + \frac{\delta}{1-\delta}\right](1-\rho)x^d}_{\text{Demand for reserves \& bonds}} = \underbrace{v}_{\text{Supply}}. \tag{29}$$

The first term on the left-hand side captures the demand for currency that facilitates production and consumption in currency-based DM transactions, while the second term represents the demand for interest-bearing assets that support deposit-based DM transactions. This equation illustrates that, in equilibrium, the aggregate demand for collateralizable assets, including currency, must equal their supply on the right-hand side.

A necessary condition for buyers not to invest in theft technology is given by

$$\varepsilon \geq \frac{\rho(x^c+\mu)}{\beta}. \tag{30}$$

Finally, the no-arbitrage condition can be written as

$$R \geq \max\left\{\frac{1}{\eta+\beta\gamma}, \frac{\eta}{(\eta+\beta\gamma)[(1-\delta)u'(x^d)+\delta]}\right\} \equiv ELB. \tag{31}$$

The first argument in the max operator represents the rate of return on holding currency for banks, while the second represents its rate of return for sellers. The effectiveness of a non-par exchange rate policy depends on which argument governs the ELB.

Given the equilibrium conditions outlined above, the model can be solved as follows. First, equations (27) and (29) are used to solve for the consumption quantities (x^c, x^d), given monetary policy (R^m, η) and fiscal policy v. Then, equation (26) determines the inflation rate π, while equation (28) solves for real interest rates r^m and r^b. Inequalities (30) and (31) provide necessary conditions for this equilibrium to exist.

Effects of Monetary Policies The first finding is that the size of the central bank's balance sheet ω is irrelevant to asset prices or consumption quantities. Adjusting the balance sheet size involves the central bank's swaps between reserves and government bonds, which affect the composition of assets held by private banks. However, because reserves and bonds are perfect substitutes at the margin, this adjustment does not influence equilibrium outcomes beyond altering the asset composition.[22]

In what follows, I first examine the effects of central bank interventions through either the nominal interest rate on reserves R^m or the non-par exchange rate η, focusing on cases where both R^m and η are sufficiently close to one. I then analyze the conditions under which a non-par exchange rate policy can lower the ELB.

Increasing the Interest Rate on Reserves R^m—With the non-par exchange rate η held constant, an increase in R^m encourages private banks to hold more interest-bearing assets, such as reserves and government bonds, rather than currency. This portfolio rebalancing increases the stock of collateral held by banks, thus relaxing their collateral constraints. As a result, banks can offer a larger quantity of deposit claims to buyers, leading to an increase in the consumption quantity in deposit-based DM transactions x^d. However, as banks reduce their currency holdings, they offer less currency to buyers, leading to a decrease in the consumption quantity in currency-based DM transactions x^c.

From (26), the decrease in currency demand is accompanied by a rise in the inflation rate π, or equivalently, a fall in the real rate of return on currency. Also, from (28), the increase in the effective stock of collateral held by banks reduces the liquidity premium on collateralizable assets, leading to an increase in real interest rates r^m and r^b, albeit to a lesser

[22]Central bank balance sheet expansions matter when reserves and government bonds are imperfect substitutes. For example, Williamson (2016) considers differences in maturity structure to show how the central bank's swaps of short-term reserves for long-term government bonds can mitigate the collateral constraints faced by banks. In contrast, Williamson (2019) and Kim (2024) focus on legal restrictions on reserve accounts and varying levels of financial regulation across institutions to show the adverse effects of central bank balance sheet expansions.

For a sufficiently low v

	∂x^c	∂x^d	$\partial \pi$	∂r^m	∂ELB
∂R^m	$-$	$+$	$+$	$+$	$\cdot$
$\partial \eta$	$+$	$-$	$+$	$-$	$-$

For a sufficiently high v

	∂x^c	∂x^d	$\partial \pi$	∂r^m	∂ELB
∂R^m	$-$	$+$	$+$	$+$	$\cdot$
$\partial \eta$	$+$	$+$	$-$	$+$	$-$

Table 1: Effects of monetary policy (R^m, η) in an equilibrium with no theft

extent than the increase in nominal interest rates.

Increasing the Non-Par Exchange Rate η—An interesting finding is that *introducing a non-par exchange rate for currency withdrawals itself has real effects.* This result is novel because a non-par exchange rate policy is commonly viewed as neutral aside from its impact on the ELB. Specifically, with R^m held constant, an increase in η raises the consumption quantity in currency-based DM transactions x^c. This occurs as a higher η increases the price of currency in the CM, thereby increasing its value as a means of payment in the DM.

The effect of an increase in η on the consumption quantity in deposit-based DM transactions x^d depends on the level of consolidated government debt v. If v is sufficiently low, an increase in currency holdings reduces the stock of collateral, leading to a decrease in x^d. Conversely, if v is sufficiently high—but not too high—an increase in η reduces the demand for currency, as the income effect outweighs the substitution effect. As a result, the stock of collateral held by banks increases, which results in an increase in x^d. These findings are formalized by the following proposition and summarized in Table 1.

Proposition 1 (Comparative Statics - No Theft) *Suppose (30) and (31) hold,* (R^m, η) *is sufficiently close to* $(1, 1)$*, and* μ *is sufficiently close to zero. Then,* $\frac{dx^c}{dR^m} < 0$, $\frac{dx^d}{dR^m} > 0$, $\frac{dr}{dR^m} > 0$, $\frac{d\pi}{dR^m} > 0$, *and* $\frac{d(ELB)}{dR^m} = 0$. *Moreover, there exists a threshold* $\hat{v}$ *such that:*
i) for $v \in (0, \hat{v}]$, $\frac{dx^c}{d\eta} > 0$, $\frac{dx^d}{d\eta} < 0$, $\frac{dr}{d\eta} < 0$, $\frac{d\pi}{d\eta} > 0$, *and* $\frac{d(ELB)}{d\eta} < 0$, *and*
ii) for $v \in (\hat{v}, \bar{v}]$, $\frac{dx^c}{d\eta} > 0$, $\frac{dx^d}{d\eta} > 0$, $\frac{dr}{d\eta} > 0$, $\frac{d\pi}{d\eta} < 0$, *and* $\frac{d(ELB)}{d\eta} < 0$, *where* $\bar{v}$ *denotes the upper bound on* v *that supports an equilibrium with a binding collateral constraint.*

The Effective Lower Bound—If the non-par exchange rate for currency withdrawals η is

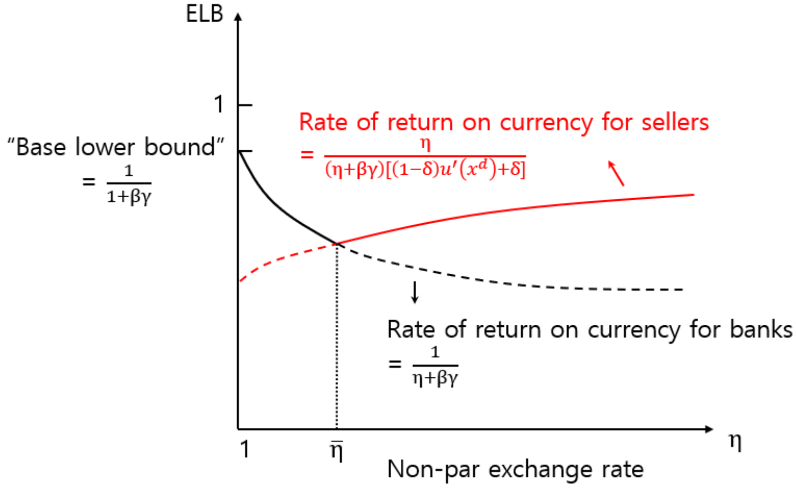

Figure 2: Exchange rate η and the effective lower bound (ELB)

sufficiently close to one, the ELB is determined by the first argument in (31). This implies that if private banks do not face arbitrage opportunities from holding currency across periods, sellers also lack such opportunities. In this case, the central bank can lower the ELB below the *base lower bound*—the level of ELB under a conventional one-to-one exchange rate between currency and reserves ($\eta = 1$).

However, if η is sufficiently high, the second argument in (31) instead determines the ELB. In this case, the lack of arbitrage opportunities for private banks does not necessarily prevent arbitrage activities for sellers. Sellers can purchase currency at price η and resell it in the next period at the same price without facing the threat of theft. This implies that an increase in η may not effectively reduce the rate of return on currency perceived by sellers, as the cost of storing currency relative to its benefit decreases with the rise in η. Essentially, the marginal cost of holding currency does not rise enough to offset the increase in its marginal benefit. Consequently, *an increase in η does not necessarily result in a lower ELB*. The following corollary states this relationship between the non-par exchange rate η and the ELB, which is also illustrated in Figure 2.

Corollary 1 (Threshold for the Non-Par Exchange Rate) *There exists a threshold $\bar{\eta} > 1$ below which the first argument in (31) determines the ELB on nominal interest rates, and above which the second argument governs it.*

Therefore, for a non-par exchange rate policy to be effective, agents must continue to deposit and withdraw currency at different exchange rates in equilibrium. If either of these activities is absent, the non-par exchange rate may fail to reduce the rate of return on currency, and thereby fail to lower the ELB. In the equilibrium examined here, the absence of currency deposits by sellers disrupts the relationship between the non-par exchange rate and the nominal rate of return on currency.[23]

Policy Combination—If the central bank aims to reduce the nominal interest rate R^m below the base lower bound, it can do so by adjusting the ELB. The following corollary examines the effects of a policy in which the central bank lowers R^m by reducing the ELB by the same magnitude.

Corollary 2 (Policy Combination - No Theft) *Given that (R^m, η) is sufficiently close to $(1, 1)$ and μ is sufficiently close to zero, consider a policy in which the central bank increases η and decreases R^m while keeping ηR^m constant. This policy leads to in an increase in x^c and a decrease in the ELB. Furthermore,*
i) for $v \in (0, \hat{v}]$, this policy decreases x^d, r^m, and r^b, while its effect on π is ambiguous, and
ii) for $v \in (\hat{v}, \bar{v})$, it reduces π, while its effects on x^d, r^m, and r^b are ambiguous.

Although this policy effectively reduces the ELB, it encourages currency side-trades in the CM, minimizing currency withdrawals from the central bank's cash window. As a result, currency becomes more valuable in the DM, leading to an increase in x^c. However, its effects on other variables—such as x^d, r^m, r^b, and π—depend on the level of consolidated government debt v. Importantly, these real effects arise because the expected payoff for

[23] Conversely, in the presence of deflation, it is possible that private banks may opt not to withdraw currency from the central bank, as discussed in Appendix A.2.

sellers from side-trading currency with private banks exceeds the payoff from depositing it with the central bank. Consequently, the expected rate of return on currency perceived by sellers deviates from the target rate set by the central bank, distorting the allocation of currency and deposit claims in DM transactions.

3.4.2 Equilibrium with Theft

Here, I examine an equilibrium where sellers sometimes carry currency into the TS, and buyers sometimes steal it. This type of equilibrium arises when the effort cost of theft ε is sufficiently low to promote the acquisition of theft technology. In this equilibrium, a non-par exchange rate is always effective in lowering the ELB because the marginal cost of carrying currency into the TS is endogenously determined by the risk of theft.

Suppose ε is sufficiently low to generate $\alpha^b \in (0, 1)$ and $\alpha^s \in (0, 1)$ in equilibrium. Then, from the first-order conditions of the bank's problem and Lemma 1, the inflation rate π and nominal interest rates R^m and R^b can be expressed as

$$\pi = \frac{\beta}{\eta}\left[u'(x^c) - \delta u'(x^d) + \delta\right], \tag{32}$$

$$R = R^m = R^b = \frac{u'(x^c) - \delta u'(x^d) + \delta}{\eta\left[u'(x^d) - \delta u'(x^d) + \delta\right]}, \tag{33}$$

$$r = r^m = r^b = \frac{1}{\beta\left[u'(x^d) - \delta u'(x^d) + \delta\right]}. \tag{34}$$

Similar to the no-theft equilibrium, a binding collateral constraint causes x^d (the consumption quantity in deposit-based DM transactions) to be inefficiently low—that is, $u'(x^d) > 1$—which translates into low real interest rates.

From (13)-(18) and the first-order conditions, the binding collateral constraint (11) can be rewritten as

$$\underbrace{\left[u'(x^c)+\frac{\delta}{1-\delta}\right]\rho(x^c+\mu)}_{\text{Demand for currency}} + \underbrace{\left[u'\left(x^d\right)+\frac{\delta}{1-\delta}\right](1-\rho)x^d}_{\text{Demand for reserves \& bonds}} = \underbrace{v}_{\text{Supply}}. \tag{35}$$

Similar to equation (29), the above equation ensures that the aggregate demand for currency and other collateralizable assets equals their aggregate supply in equilibrium.

From Lemma 1, the fraction of buyers who acquire theft technology α^b and the fraction of sellers who carry currency in the TS α^s can be expressed as

$$\alpha^b = \frac{\eta-1}{\eta}; \qquad \alpha^s = \frac{\beta\varepsilon}{\rho\eta(x^c+\mu)}, \tag{36}$$

while a necessary condition for buyers to invest in theft technology can be rewritten as

$$\varepsilon < \frac{\rho\eta(x^c+\mu)}{\beta}. \tag{37}$$

Finally, the no-arbitrage condition related to storing currency can be written as

$$R^m \geq \frac{1}{\eta+\beta\gamma} \equiv ELB. \tag{38}$$

Solving the model is straightforward, as in an equilibrium without theft. First, equations (33) and (35) determine the consumption quantities (x^c, x^d), given monetary policy (R^m, η) and fiscal policy v. Then, equation (32) solves for the inflation rate π, while equation (34) determines real interest rates r^m and r^b. Equations in (36) determine α^b and α^s, and inequalities (37) and (38) provide necessary conditions for this equilibrium to exist.

Effects of Monetary Policies The size of the central bank's balance sheet ω remains irrelevant to asset prices or consumption quantities. Therefore, I examine the effects of monetary policy interventions, focusing on R^m (the nominal interest rate on reserves) and η

(the non-par exchange rate for currency withdrawals).

Increasing the Interest Rate on Reserves R^m—With η held constant, the effects of an increase in the nominal interest rate R^m on consumption quantities (x^c, x^d), real interest rates (r^m, r^b), and the inflation rate π are qualitatively identical to those in an equilibrium without theft. However, in this equilibrium, the fraction of sellers who carry currency into the TS α^s increases. This happens because, in response to the increase in R^m, sellers receive a smaller quantity of currency from buyers in the DM (i.e., c' decreases). Due to the decrease in currency carried by sellers, buyers are less inclined to invest in theft technology. Consequently, sellers can increase the probability of carrying currency into the TS until the fraction of buyers investing in theft technology α^b returns to its previous level.

Increasing the Non-Par Exchange Rate η—An increase in the non-par exchange rate η, which raises the price of currency in the CM, not only induces sellers to carry more currency into the TS but also encourages more buyers to invest in theft technology. As the marginal cost of currency side-trading rises endogenously to offset its marginal benefit, the rate of return on currency perceived by sellers aligns with the central bank's target rate. As a result, *the central bank can effectively reduce the rate of return on currency and the ELB without a bound.*

Also, an increase in η leads to a decrease in the inflation rate π, which tends to raise the real return on currency. Thus, the fall in the real rate of return on currency due to an increase in η is partially offset by the decrease in π in equilibrium. These findings are consistent with the arguments of Eisler (1932), Buiter (2010), and Agarwal and Kimball (2015), who suggest that introducing a non-par exchange rate can lower the nominal rate of return on currency and, in turn, the ELB. However, *reducing the ELB comes at a cost*, as a larger fraction of buyers investing in theft technology necessitates allocating more resources to this activity, which merely redistributes assets without improving social welfare.

Similar to what happens in an equilibrium without theft, an increase in the non-par exchange rate η itself has real effects. Specifically, it decreases x^c and increases x^d, r^m and

	∂x^c	∂x^d	$\partial \pi$	∂r^m	∂ELB	$\partial \alpha^b$	$\partial \alpha^s$
∂R^m	$-$	$+$	$+$	$+$	$\cdot$	$\cdot$	$+$
$\partial \eta$	$-$	$+$	$-$	$+$	$-$	$+$	?

Table 2: Effects of monetary policy (R^m, η) in an equilibrium with theft

r^b. These effects arise because, with R^m held constant, an increase in η leads to a fall in the rate of return on currency relative to government bonds and reserves $\frac{1}{\eta R^m}$. This implies that the effects of an increase in η on consumption quantities and real interest rates are identical to those of an increase in R^m. These results are formalized by the following proposition and summarized in Table 2.

Proposition 2 (Comparative Statics - Theft) *Suppose that inequalities (37) and (38) hold. Then,* $\frac{dx^c}{dR^m} < 0$, $\frac{dx^d}{dR^m} > 0$, $\frac{dr}{dR^m} > 0$, $\frac{d\pi}{dR^m} > 0$, $\frac{d\alpha^b}{dR^m} = 0$, $\frac{d\alpha^s}{dR^m} > 0$, *and* $\frac{d(ELB)}{dR^m} = 0$. *In addition,* $\frac{dx^c}{d\eta} < 0$, $\frac{dx^d}{d\eta} > 0$, $\frac{dr}{d\eta} > 0$, $\frac{d\pi}{d\eta} < 0$, $\frac{d\alpha^b}{d\eta} > 0$, *and* $\frac{d(ELB)}{d\eta} < 0$, *while* $\frac{d\alpha^s}{d\eta}$ *is ambiguous.*

Policy Combination—If the central bank seeks to set the nominal interest rate R^m below the base lower bound, it must lower the ELB on nominal interest rates. Suppose the central bank increases η and reduces R^m by the same magnitude, to implement R^m below the base lower bound. This policy leads to a decrease in the inflation rate π one-for-one with an increase in η, leaving the real rate of return on currency unchanged. Also, real interest rates (r^m, r^b) remain unaffected, as the decrease in π exactly offsets the decrease in R^m (i.e., there arises a pure Fisher effect). Consequently, *this policy yields no welfare gain*, as the consumption quantities (x^c, x^d) remain unchanged. However, it increases the fraction of buyers who acquire theft technology, which has negative welfare implications.

Corollary 3 (Policy Combination - Theft) *Consider a policy in which the central bank increases η and decreases R^m while keeping ηR^m constant. This policy decreases π one-for-one, decreases α^s and the ELB, and increases α^b. However, consumption quantities (x^c, x^d) and real interest rates (r^m, r^b) remain unchanged.*

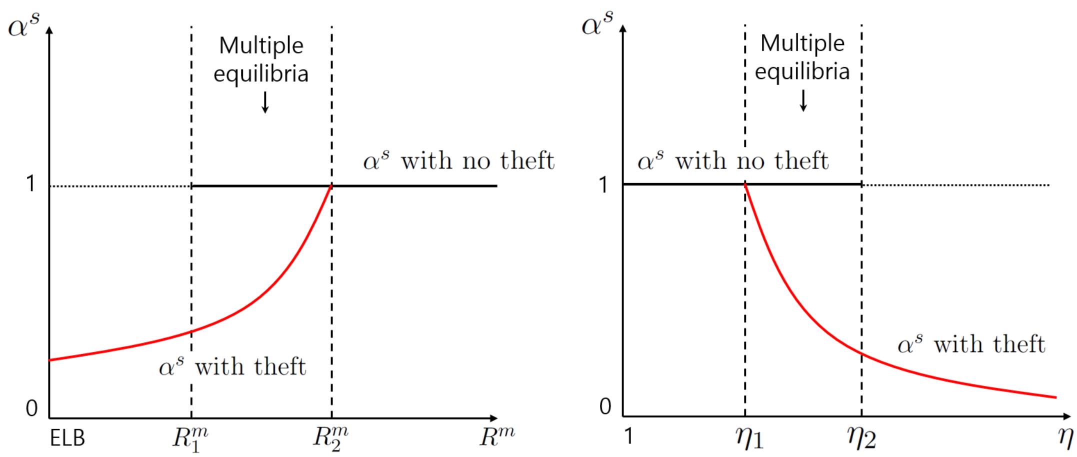

Figure 3: Equilibrium α^s with monetary policy (R^m, η)

3.4.3 Monetary Policies and the Equilibrium Type

With the necessary conditions (30) and (37), Propositions 1 and 2 help to clarify the type of equilibrium that may arise under different monetary policies (R^m, η). When η remains constant, a decrease in R^m increases x^c, making it more likely for an equilibrium with theft to emerge. Specifically, if η is sufficiently high, or if the cost of theft ε is sufficiently low, theft occurs when R^m is sufficiently low. In such a case, there may be a range of nominal interest rates R^m that support equilibria both with and without theft, as illustrated in the left panel of Figure 3. In the figure, R_2^m, the upper bound for nominal interest rates supporting the *theft* equilibrium, is higher than R_1^m, the lower bound for rates supporting the *no-theft* equilibrium. Hence, for any $R^m \in [R_1^m, R_2^m]$, multiple equilibria emerge. However, for some parameter values, the upper bound R_2^m can be lower than the lower bound R_1^m, implying the absence of equilibrium for $R^m \in (R_2^m, R_1^m)$.

Propositions 1 and 2 also state that, with R^m held constant, an increase in η decreases x^c in a theft equilibrium while increasing x^c in a no-theft equilibrium. This implies that a no-theft equilibrium is more likely to emerge when η is lower and ε (the cost of theft) is higher. Specifically, if either R^m or ε is sufficiently high, theft does not occur for sufficiently low values of η. In such cases, multiple equilibria may exist for values of η that are sufficiently

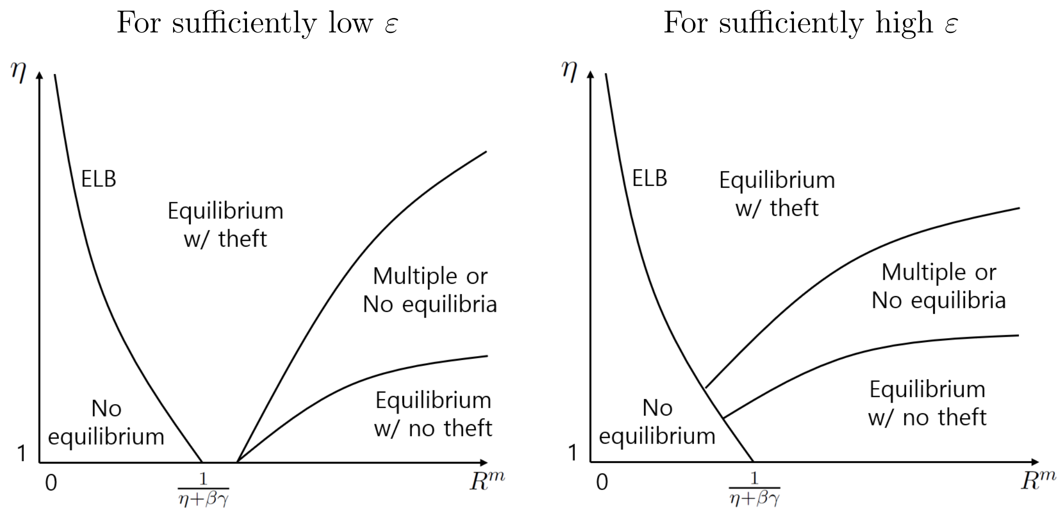

Figure 4: Equilibria with monetary policy (R^m, η)

high but not too high, as illustrated in the right panel of Figure 3. Although the effect of an increase in η on α^s (the fraction of sellers carrying currency into the TS) is ambiguous in a theft equilibrium, it can be shown that $\alpha^s \to 0$ as $\eta \to \infty$. Lastly, Figure 4 illustrates how monetary policies (R^m, η) determine the existence of particular equilibria.

4 Optimal Monetary Policy

The relevant measure of welfare is the sum of surpluses from trade in the CM and the DM, net of the total cost incurred in the TS:

$$\mathcal{W} = \underbrace{0}_{\text{CM surpluses}} + \underbrace{\rho\left[u(x^c) - x^c\right] + (1-\rho)\left[u(x^d) - x^d\right]}_{\text{DM surpluses}} - \underbrace{\alpha^b \varepsilon}_{\text{total cost in TS}}. \tag{39}$$

This welfare measure is also equivalent to the sum of period utilities across agents in equilibrium. I will use this welfare measure to analyze the optimal monetary policy.

Proposition 3 (Optimal Monetary Policy - Theft) *If ε is sufficiently low and μ is sufficiently close to zero, the optimal monetary policy consists of $\eta = 1$ and $R^m \leq 1$ that* *satisfies (38).*

If the cost of theft ε is sufficiently low, it is optimal to use the conventional one-to-one exchange rate ($\eta = 1$) rather than introducing a non-par rate. To understand the intuition behind this result, consider the benchmark case where the fixed cost of holding currency in the DM is zero (i.e., $\mu = 0$). The formal proof of this result consists of two steps. First, given α^b (the fraction of buyers acquiring theft technology), the optimal monetary policy follows a *modified* Friedman rule: the nominal interest rate on reserves relative to currency is zero at the optimum, or equivalently, $\eta R^m = 1$. This rule allows buyers to perfectly smooth their consumption across DM transactions, maximizing the expected surplus from trade. Second, among the policies consistent with this rule, the welfare-maximizing one is $\eta = 1$ and $R^m = 1$. This policy combination minimizes α^b, thereby reducing the resource costs associated with socially useless theft technology while preserving trade surpluses.

If the fixed cost of holding currency μ is positive, a negative nominal interest rate ($R^m < 1$) becomes optimal under a conventional one-to-one exchange rate. This is because buyers must bring more currency into the DM to compensate sellers for their currency-holding costs. These costs distort the allocation of currency and deposit claims, creating inefficiencies in DM transactions. A negative nominal interest rate helps mitigate these inefficiencies by indirectly reducing the currency-holding costs, thereby improving welfare.[24]

Importantly, even if the optimal interest rate is constrained by the base lower bound (the ELB level under $\eta = 1$), introducing a non-par exchange rate does not enhance welfare when ε is sufficiently low.[25] This is because reducing the ELB through a non-par exchange rate has real effects that offset the benefits of lowering the nominal interest rate R^m. Specifically, if the central bank reduces both the ELB and R^m by the same magnitude, the nominal interest rate relative to the rate of return on currency remains unchanged, yielding no welfare gains. Since this policy merely increases welfare costs through greater investment in theft technology, it

[24] See Appendix A.5 for formal proof.

[25] In other words, this new policy does not expand the set of implementable allocations in this economy when ε is sufficiently low.

remains optimal to set the nominal interest rate at the base lower bound.

Proposition 4 (Optimal Monetary Policy - No Theft) *If ε is sufficiently high, the optimal monetary policy is given by $\eta = \tilde{\eta}(R^m)$, where $\tilde{\eta}(R^m)$ is the solution to (27) and (29) for $x^c = \beta(\frac{\varepsilon}{\rho} - \mu)$ and R^m satisfying (31). Moreover, if μ is sufficiently close to zero, the optimal R^m satisfies $R^m > 1/\tilde{\eta}$.*

Interestingly, if the cost of theft ε is sufficiently high to prevent theft in equilibrium, the optimal nominal interest rate can be higher than the level that satisfies a modified Friedman rule. In Appendix A.5, I show that deviating from the modified Friedman rule by increasing either the nominal interest rate R^m or the non-par exchange rate η can improve social welfare, i.e., the optimal policy can be characterized by $\eta R^m > 1$. An increase in R^m enhances welfare by enabling buyers to smooth their consumption across different DM transactions.

However, increasing η from the modified Friedman rule enhances welfare for a non-standard reason. In a no-theft equilibrium, an increase in η does not effectively reduce the perceived nominal rate of return on currency for sellers, while raising the price of currency in the CM. This increases the value of currency in the DM, leading to an increase in consumption in currency-based DM transactions x^c. Although this may be accompanied by a decrease in consumption in deposit-based DM transactions x^d, the potential decrease in the buyers' utility from the change in x^d is smaller than the increase in their utility from the increase in x^c. Therefore, an increase in η can improve welfare even if it does not smooth out consumption across different DM transactions. Furthermore, as welfare increases with the non-par exchange rate η, it is optimal to set η at the highest possible level that does not induce theft in equilibrium.

These results are visually represented by Figure 5. Each panel in the figure illustrates the Highest Welfare Locus (HWL), which shows the combinations of (R^m, η) that yield the highest welfare, assuming that no theft occurs in equilibrium. In the left panel, where the cost of theft ε is sufficiently low, welfare is maximized at $\eta = 1$ and $R^m < 1$. Conversely, in

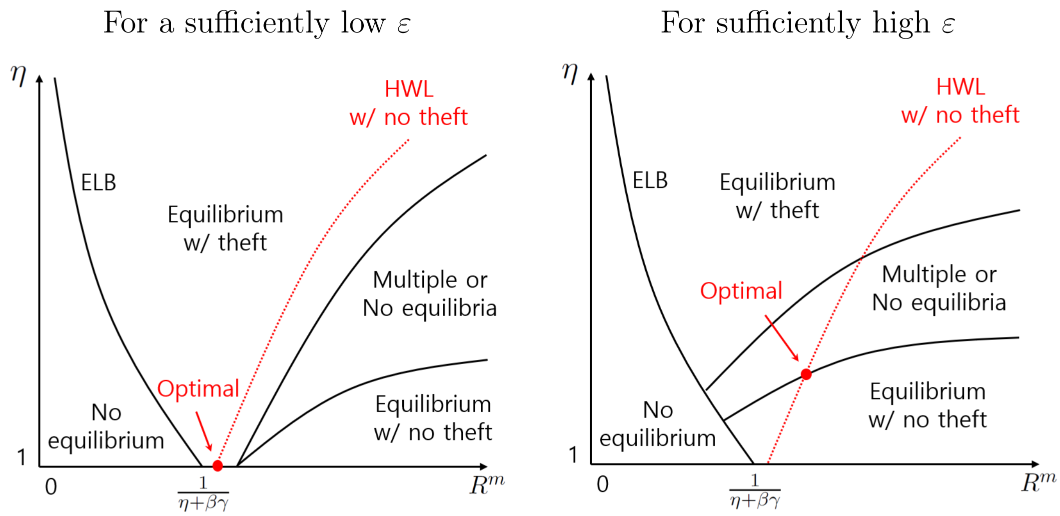

Figure 5: Optimal monetary policy

the right panel, where ε is sufficiently high, welfare is maximized at the highest possible level of η along with the corresponding R^m on the HWL that supports a no-theft equilibrium as the unique equilibrium.

5 Disintermediation

Another concern surrounding the implementation of negative nominal interest rates is the possibility of disintermediation, where consumers withdraw all their deposits in favor of holding currency.[26] In the baseline model analyzed in previous sections, disintermediation does not occur because currency and bank deposits are not substitutes. To examine how a non-par exchange rate policy helps prevent disintermediation, I extend the model by allowing buyers to use currency in all DM meetings.

Specifically, with probability ρ, buyers must use currency, whereas with probability $1-\rho$, they can choose between currency and deposit claims. In this setting, some buyers may find it optimal to opt out of banking arrangements. Let θ denote the fraction of buyers who choose

[26] Disintermediation is indeed a practical concern. As a negative deposit rate might lead to massive cash withdrawals, private banks are reluctant to reduce their deposit rates below zero. See Eggertsson, Juelsrud, Summers, and Wold (2022) for empirical evidence on the breakdown of the monetary policy transmission when the policy rate is negative.

to deposit with private banks. Each private bank's contracting problem remains identical to the one in the baseline model. Meanwhile, the fraction $1-\theta$ of buyers opt out of banking arrangements and choose the optimal quantity of currency to use in all DM meetings.

Interestingly, if v (the real value of consolidated government debt outstanding) is sufficiently low, equilibrium exists only if the nominal interest rate R^m is sufficiently high to support banking activities. In general, a fall in R^m increases the demand for currency both intensively and extensively, forcing the central bank to issue more currency through open market purchases. However, in a complete disintermediation scenario, the central bank would be unable to meet public demand for currency, as the scale of its open market purchases is limited by the supply of government bonds. Consequently, complete disintermediation cannot be sustained in equilibrium, creating an additional lower bound on R^m.[27]

I also find that if the cost of theft is sufficiently low, the nominal interest rates that support banking activities are sufficiently high to prevent arbitrage opportunities from storing currency across periods. In other words, the effective lower bound (ELB) on R^m is determined by the lower bound on disintermediation-free interest rates rather than the lower bound on arbitrage-free interest rates. Nevertheless, introducing a non-par exchange rate η is still effective in reducing the ELB.

Lowering R^m contributes to disintermediation by incentivizing more buyers to opt out of banking arrangements. The central bank can offset this effect by increasing η, as this helps maintain the relative rate of return on currency at the previous level. Thus, a non-par exchange rate enables the central bank to implement negative interest rates without triggering disintermediation. Nonetheless, this policy leads only to welfare losses—as in the baseline model—by encouraging currency side-trading and theft.[28]

[27] This finding is related to Eggertsson, Juelsrud, Summers, and Wold (2022), who show the coexistence of the lower bound on deposit rates (disintermediation-free interest rates) and the lower bound on short-term policy rates (arbitrage-free interest rates).

[28] See Appendix A.3 for details.

6 Conclusion

A non-par exchange rate between currency and reserves has been proposed as a potential policy tool for reducing the effective lower bound (ELB) on nominal interest rates. To examine the implications of introducing this new policy, I have developed a model with multiple assets, incorporating frictions associated with holding currency. A key finding is that a non-par exchange rate can indeed reduce the ELB, as these frictions incentivize economic agents to maintain their deposit accounts rather than avoiding the banking system.

However, introducing a non-par exchange rate may entail welfare costs by intensifying the frictions that determine the ELB. Specifically, this policy can raise the market value of currency, promoting socially undesirable behaviors like currency side-trading and theft, ultimately leading to welfare losses. Importantly, even if the optimal interest rate is constrained by the current level of lower bound, introducing a non-par exchange rate may not help increase welfare, as the effects of reducing the ELB can offset the effects of lowering the nominal interest rate. Consequently, the optimal policy is to set the nominal interest rate at the current ELB level while maintaining the one-to-one exchange rate between currency and reserves to avoid potential welfare losses.

Replacing physical currency with central bank digital currency (CBDC) can also help reduce the ELB. This is because the central bank can directly introduce a negative nominal interest rate on CBDC, a possibility not feasible with physical currency. However, while the central bank can easily reduce the ELB, the economic effects of implementing a negative interest rate on reserves along with a negative interest rate on CBDC remain uncertain. The findings of this paper suggest that such a policy combination may not improve welfare, as the nominal interest rate on reserves relative to CBDC, which determines the monetary policy stance, would remain unchanged.[29]

[29] This conjecture aligns with the findings of Williamson (2022), which demonstrate that in an economy with CBDC, government bonds, and private capital, the relevant interest rate is the interest rate on government bonds relative to CBDC rather than the interest rate on government bonds relative to zero-interest currency.

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A Appendix

A.1 First-Order Conditions for Private Bank's Problem

Let λ denote the Lagrange multiplier associated with the collateral constraint (11). Then, the first-order conditions for the bank's problem (9), subject to (10)-(12), are given by

$$\frac{\beta[1-\alpha^s+\alpha^s(1-\alpha^b)\eta]}{\pi}u'\left([1-\alpha^s+\alpha^s(1-\alpha^b)\eta]\frac{\beta c'}{\pi}-\mu\right)-\eta-\frac{\lambda\delta}{\pi}=0, \tag{40}$$

$$\beta u'(\beta d)-\beta-\lambda=0, \tag{41}$$

$$-1+\frac{\beta R^m}{\pi}+\frac{\lambda R^m(1-\delta)}{\pi}=0, \tag{42}$$

$$-1+\frac{\beta R^b}{\pi}+\frac{\lambda R^b(1-\delta)}{\pi}=0, \tag{43}$$

$$-\eta+\frac{\beta}{\pi}-\beta\gamma+\frac{\lambda(1-\delta)}{\pi}\leq 0, \tag{44}$$

$$\lambda\left[-(1-\rho)d+\frac{(1-\delta)\left(R^m m+R^b b+c\right)}{\pi}-\frac{\rho c'}{\pi}\right]=0. \tag{45}$$

A.2 Equilibrium with Deflation and No Theft

In this section, I focus on stationary equilibria characterized by deflation and a sufficiently high cost of theft ε. With deflation, the real value of currency increases over time, allowing private banks to acquire a sufficient amount of currency in the CM. As private banks do not need to withdraw currency from the central bank, the price of currency is not determined by the non-par exchange rate η. In this case, sellers become indifferent between depositing their currency with the central bank and engaging in side-trading with private banks. Thus, in equilibrium, one unit of real currency is exchanged for one unit of goods in the CM. From (7) and Lemma 1, a necessary condition for this equilibrium to exist is given by

$$\varepsilon\geq\frac{\rho(x^c+\mu)}{\beta}.$$

As the price of real currency is one, the values of η in equilibrium conditions (13)-(17)

and (40)-(45) must be replaced by one. Then, from (25), the effective lower bound (ELB) on nominal interest rates R^m can be defined as

$$R^m \geq \frac{1}{1+\beta\gamma} \equiv ELB.$$

This implies that the non-par exchange rate η is irrelevant to the ELB. The non-par exchange rate policy is ineffective because, if it is introduced, private banks would stop withdrawing currency from the central bank. Also, it is obvious that the non-par exchange rate would not affect equilibrium prices and allocations.

The inflation rate in this equilibrium can be written as

$$\pi = \beta \left[u'(x^c) - \delta u'(x^d) + \delta \right].$$

To observe deflation in equilibrium, both β and $u'(x^c)$ must be sufficiently low, while δ and $u'(x^d)$ must be sufficiently high. It turns out that υ must be sufficiently high and R^m must be sufficiently low to support deflation in equilibrium. Note that, in the main body of the paper, I focus on cases with sufficiently low υ and γ to ensure that there is inflation in equilibrium for any R^m higher than the ELB.

A.3 Equilibrium with Potential Disintermediation

In this section, I formally characterize the extended model introduced in Section 5 on Disintermediation. In Sections 2-4, I assumed that a fraction ρ of buyers must use currency, while the remaining buyers must use non-currency assets. In contrast, the extended model allows buyers to opt out of banking arrangements and use currency in all DM transactions. This implies that the fraction $1-\rho$ of buyers can choose between currency and bank deposits, while the fraction ρ continues to use currency as in the baseline model. Apart from this modification, the model remains the same as described in Section 2.

A.3.1 Optimization

In this setting, some buyers may find it optimal to opt out of banking arrangements. Let θ denote the fraction of buyers who choose to deposit with private banks in the CM. Each bank's contracting problem and first-order conditions remain identical to those in the baseline model. Meanwhile, the fraction $1-\theta$ of buyers opt out of banking arrangements and choose the optimal quantity of currency c^o to solve the following problem:

$$\max_{c^o \geq 0} \left\{ -\eta c^o + u\left(\frac{\beta c^o \left[1-\alpha^s+\alpha^s(1-\alpha^b)\eta\right]}{\pi} - \mu \right) \right\}. \tag{46}$$

Although banks do not write deposit contracts with these buyers, they withdraw currency from the central bank to sell it to these buyers whenever necessary. With inflation, the price of currency in equilibrium is η, as in the baseline model. The first-order condition of the above problem is given by

$$-\eta + \frac{\beta\left[1-\alpha^s+\alpha^s(1-\alpha^b)\eta\right]}{\pi} u'\left(\frac{\beta c^o \left[1-\alpha^s+\alpha^s(1-\alpha^b)\eta\right]}{\pi} - \mu \right) = 0. \tag{47}$$

Let x^o denote the consumption quantity in the DM for these non-contracting buyers. The asset market clearing conditions are

$$\theta c + (1-\theta)c^o = \bar{c}; \qquad \theta m = \bar{m}; \qquad \theta b = \bar{b}. \tag{48}$$

Let U^b and U^o denote the expected utilities for buyers who obtain banking contracts and those who opt out, respectively. Using the first-order conditions for the bank's problem and equations (10) and (47), I obtain

$$U^b = \rho\left[u(x^c) - (x^c+\beta\mu)u'(x^c)\right] + (1-\rho)\left[u(x^d) - x^d u'(x^d)\right], \tag{49}$$

$$U^o = u(x^o) - (x^o+\beta\mu)u'(x^o). \tag{50}$$

In equilibrium, the fraction θ must solve the following problem:

$$\max_{0\leq\theta\leq 1}\left[\theta U^b + (1-\theta)U^o\right], \tag{51}$$

which implies that buyers' participation in banking arrangements, represented by θ, must be consistent with their utility maximization problem.

Depending on whom each seller met in the previous DM, the quantity of currency holdings in the TS varies across sellers. Buyers who hold deposit contracts withdraw c' units of currency (in real terms) to purchase goods in the DM. Thus, sellers who meet these buyers in the DM will hold $\frac{c'}{\pi}$ units of currency in the following TS. Meanwhile, buyers who do not hold deposit contracts have c^o units of currency. Therefore, sellers who meet these buyers in the DM will hold $\frac{c^o}{\pi}$ units of currency in the following TS. For simplicity, I assume that each seller decides whether to carry currency into the TS before knowing the type of the buyer they will meet in the DM.

The fraction of buyers acquiring theft technology, α^b, and the fraction of sellers carrying currency into the TS, α^s, must be incentive-compatible in equilibrium:

$$\alpha^b = \begin{cases} 0 & \text{if } \varepsilon > \frac{[\theta\rho c' + (1-\theta)c^o]\alpha^s\eta}{\pi} \\ \in [0,1] & \text{if } \varepsilon = \frac{[\theta\rho c' + (1-\theta)c^o]\alpha^s\eta}{\pi} \\ 1 & \text{if } \varepsilon < \frac{[\theta\rho c' + (1-\theta)c^o]\alpha^s\eta}{\pi} \end{cases} \tag{52}$$

With probability $\theta\rho\alpha^s$, a buyer is matched with a seller holding $\frac{c'}{\pi}$ units of currency, and with probability $(1-\theta)\alpha^s$, the buyer meets a seller holding $\frac{c^o}{\pi}$ units of currency. The above condition states that theft does not occur if it is too costly, occurs sometimes if the buyer is indifferent between stealing and not stealing, and always occurs if it is strictly profitable.

Also, each seller's decision on whether to carry currency into the TS must be incentive-compatible:

$$\alpha^s = \begin{cases} 0 & \text{if } (1-\alpha^b)\eta < 1 \\ \in [0,1] & \text{if } (1-\alpha^b)\eta = 1 \\ 1 & \text{if } (1-\alpha^b)\eta > 1 \end{cases} \tag{53}$$

A.3.2 Effective Lower Bound on Nominal Interest Rates

I will focus on cases where the cost of theft ε is sufficiently small to sustain a theft equilibrium.[30] Additionally, I will consider scenarios where the supply of collateralizable assets is insufficient to support the first-best level of consumption in deposit-based DM transactions, as in previous sections. Specifically, I assume that

$$v < x^* + \mu, \tag{54}$$

where x^* is the efficient level of consumption in DM transactions that satisfies $u'(x^*) = 1$. This assumption implies that consumption in DM transactions for buyers choosing bank contracts is inefficiently low due to a shortage of collateral. Moreover, it also implies that the central bank cannot support the efficient level of consumption for buyers opting out of banking contracts. This is because the quantity of currency issued by the central bank is constrained by the size of its balance sheet, which can only expand through open market purchases.

Given this assumption, the following proposition characterizes the effective lower bound (ELB) on nominal interest rates R^m.

[30] For cases with a sufficiently high cost of theft, equilibrium conditions can be presented, but analyzing the effects of monetary policy in such cases may not be straightforward.

Proposition A.3.1 (Effective Lower Bound) *Suppose that both ε and μ are sufficiently low, and that v satisfies (54). Then, an equilibrium exists if and only if*

$$R^m \geq \frac{u'(x^o)}{\eta[(1-\delta)u'(x^d)+\delta]}, \tag{55}$$

where (x^o, x^d), together with x^c, consist of the solution to $(x^o + \beta\mu)u'(x^o) = v$, $u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta$, and $U^b = U^o$. Furthermore,

$$\frac{u'(x^o)}{(1-\delta)u'(x^d)+\delta} \geq 1.$$

Interestingly, if v (the real value of consolidated government debt outstanding) is sufficiently low, equilibrium exists only if the nominal interest rate R^m is sufficiently high to support banking activities. In general, a fall in R^m increases the demand for currency both intensively and extensively, forcing the central bank to issue more currency through open market purchases. However, in a complete disintermediation scenario (i.e., $\theta = 0$), the central bank would be unable to meet public demand for currency, as the scale of its open market purchases is limited by the supply of government bonds. Thus, complete disintermediation cannot be sustained in equilibrium, creating an additional lower bound on R^m.[31]

Another notable finding is that the ELB on nominal interest rates can even be positive. This occurs due to the shortage of collateralizable assets, which creates inefficiencies in the banking system. When the nominal interest rate is zero ($R^m = 1$), each contracting buyer would consume the same quantity of goods across different DM meetings, i.e., $x^c = x^d$. However, if the fixed cost of holding currency μ is sufficiently low, a non-contracting buyer would consume more in the DM than a contracting buyer, i.e., $x^o > x^c = x^d$, leading to complete disintermediation ($\theta = 0$). Since this outcome is not sustainable in equilibrium,

[31] This finding is related to Eggertsson, Juelsrud, Summers, and Wold (2022), who show the coexistence of the lower bound on deposit rates (disintermediation-free interest rates) and the lower bound on short-term policy rates (arbitrage-free interest rates).

the nominal interest rate must be strictly positive to support the banking system.[32]

This result implies that, given a sufficiently low μ, the nominal interest rates that support the banking system are sufficiently high to prevent arbitrage opportunities from storing currency across periods. Thus, the ELB is determined by the lower bound on disintermediation-free interest rates (the right-hand side of 55) rather than the lower bound on arbitrage-free interest rates.[33] From (55), introducing a non-par exchange rate η is still effective in lowering the ELB. However, the proportional cost of storing currency γ becomes irrelevant because storing currency across periods is never profitable.

A.3.3 Characterization of Equilibrium

Here, I focus on cases where $\mu = 0$ (no fixed cost of holding currency at the beginning of the TS) for analytical convenience.[34] The equilibrium conditions are then given by

$$\eta R^m = \frac{u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \tag{56}$$

$$(1-\rho)\theta x^d \left[u'(x^d) + \frac{\delta}{1-\delta}\right] + \rho\theta(x^c)\left[u'(x^c) + \frac{\delta}{1-\delta}\right] + (1-\theta)(x^o)u'(x^o) = v, \tag{57}$$

$$\pi = \frac{\beta\left[u'(x^c) - \delta u'(x^d) + \delta\right]}{\eta}. \tag{58}$$

Equations (56)-(58) come from the first-order conditions of a private bank's problem in equilibrium, with (57) representing the collateral constraint. Notably, this collateral constraint differs from that in the baseline model 35, as it accounts for the central bank's purchases of government bonds when issuing currency for non-contracting buyers.

[32] One can imagine that a higher fixed storage cost μ would increase inefficiencies in currency-based DM transactions. This would discourage buyers from opting out of deposit contracts, thereby lowering the ELB. While analyzing the qualitative effect of an increase in μ appears to be complicated, the intuition suggests that a sufficiently high μ would lead to a negative ELB on nominal interest rates.

[33] This finding is related to Eggertsson, Juelsrud, Summers, and Wold (2022), in that the lower bound on deposit rates (disintermediation-free interest rates) tends to be higher than the lower bound on short-term policy rates (arbitrage-free interest rates). While my model abstracts from addressing the conflict between these different lower bounds, it suggests that introducing a non-par exchange rate policy could reduce both lower bounds by decreasing the rate of return on holding currency for both banks and depositors.

[34] This assumption simplifies the analysis while still providing insights that can be extended to cases with sufficiently low $\mu > 0$.

From each buyer's problem in the CM, represented by (47) and (49)-(51), I obtain

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \tag{59}$$

$$\rho\left[u(x^c) - x^c u'(x^c)\right] + (1-\rho)\left[u(x^d) - x^d u'(x^d)\right] \geq u(x^o) - x^o u'(x^o). \tag{60}$$

Equation (59) ensures positive consumption quantities in the DM for both contracting and non-contracting buyers while also requiring that their choices be incentive-compatible in equilibrium. Since collateral scarcity constrains both x^c and x^d, buyers opt for deposit contracts only when $x^c < x^d$, as they would otherwise prefer to opt out. Moreover, if $x^o < x^c < x^d$, no buyers would opt out of deposit contracts, whereas if $x^c < x^d < x^o$, no buyers would opt in. Thus, the DM consumption quantities must satisfy $x^c < x^o < x^d$ in equilibrium, as indicated by (59).[35] Equation (60) further ensures that buyers weakly prefer opting for a deposit contract over opting out.

Finally, from (52)-(53), incentive-compatibility conditions in the TS are given by

$$\alpha^b = \frac{\eta - 1}{\eta}, \tag{61}$$

$$\alpha^s = \frac{\beta\varepsilon}{\eta\left[\theta\rho x^c + (1-\theta)x^o\right]}, \tag{62}$$

$$\varepsilon < \frac{\eta\left[\theta\rho x^c + (1-\theta)x^o\right]}{\beta}. \tag{63}$$

Equations (61) and (62) determine the fraction of buyers acquiring theft technology α^b and the fraction of sellers carrying currency into the TS α^s. Inequality (63) is a necessary condition for this equilibrium to exist.

The model can be solved differently depending on whether (60) holds with equality. If (60) holds with equality, equations (56), (59), and (60) determine x^c, x^d, and x^o given monetary policy (R^m, η) and fiscal policy v. Then, equation (57) solves for θ, equation (58)

[35] However, using only currency in the DM is also inefficient due to the fixed cost of holding currency μ. With a sufficiently high μ, buyers could choose to use deposit contracts even when $x^d \leq x^c$.

solves for π, and equations (61) and (62) solve for α^b and α^s, respectively.

On the other hand, if (60) holds with strict inequality, equations (56) and (57), with $\theta = 1$, solve for x^c and x^d. Then, equation (58) determines π, while (61) and (62) solve for α^b and α^s, respectively. Equation (59) solves for x^o, which is the would-be consumption quantity in the DM if a buyer were to opt out of banking arrangements off equilibrium.

A.3.4 Effects of Monetary Policies

The following proposition shows how the fraction θ is determined in equilibrium and how monetary policy (R^m, η) affects the equilibrium outcome depending on the value of θ.

Proposition A.3.2 (Comparative Statics) *Suppose $\mu = 0$. If ηR^m is sufficiently high to satisfy (55) but not too high, then $0 \leq \theta < 1$ in equilibrium and (60) holds with equality. In this case, $\frac{dx^c}{d(\eta R^m)} < 0$, $\frac{dx^o}{d(\eta R^m)} < 0$, $\frac{dx^d}{d(\eta R^m)} > 0$, $\frac{dr}{d(\eta R^m)} > 0$, and $\frac{d\theta}{d(\eta R^m)} > 0$. Furthermore, $\frac{d\pi}{dR^m} > 0$, $\frac{d\alpha^b}{dR^m} = 0$, $\frac{d\alpha^s}{dR^m} > 0$, $\frac{d\pi}{d\eta} < 0$, and $\frac{d\alpha^b}{d\eta} > 0$, while $\frac{d\alpha^s}{d\eta}$ is ambiguous. If ηR^m is very high, then $\theta = 1$ in equilibrium and (60) holds with strict inequality.*

With the non-par exchange rate η held constant, an increase in the nominal interest rate R^m decreases the rate of return on currency relative to reserves and government bonds. This leads to a substitution of deposit claims for currency. If $0 \leq \theta < 1$ in equilibrium, the consumption quantity in DM transactions using deposit claims x^d increases along with a rise in the fraction of contracting buyers θ, while the consumption quantities in DM transactions using currency x^c and x^o decrease. Thus, lowering R^m contributes to disintermediation by incentivizing more buyers to opt out of banking arrangements, i.e., by decreasing θ.

The central bank can offset this effect by increasing the non-par exchange rate η, as this helps maintain the relative rate of return on currency $1/\eta R^m$ at the previous level. If ηR^m is held constant, the fraction θ as well as the consumption quantities x^c, x^d, and x^o would remain unchanged. As a result, a non-par exchange rate enables the central bank to implement negative nominal interest rates without triggering disintermediation. However, this policy leads to welfare losses, similar to the baseline model, as it increases the fraction of buyers investing in the costly theft technology α^b.

A.3.5 Optimal Monetary Policy

The welfare measure for this economy is defined as

$$\mathcal{W} = \rho\theta\left[u\left(x^c\right) - x^c\right] + (1-\rho)\,\theta[u(x^d) - x^d] + (1-\theta)\left[u\left(x^o\right) - x^o\right] - \alpha^b\varepsilon. \qquad (64)$$

which is the sum of surpluses from trade in the CM and the DM, net of the total cost incurred in the TS. Then, the following proposition characterizes the optimal monetary policy.

Proposition A.3.3 (Optimal Monetary Policy) *Suppose $\mu = 0$. Then, the optimal monetary policy consists of $\eta = 1$ and $R^m = \frac{u'(x^o)}{u'(x^d)-\delta u'(x^d)+\delta} > 1$, where (x^o, x^d), together with x^c, consist of the solution to $x^o u'(x^o) = v$, (59), and (60) with equality.*

The central bank can optimize the surplus from trade by choosing an appropriate policy combination of (R^m, η). While there exists a set of such combinations that achieve the maximized level of trade surplus, maintaining the one-to-one exchange rate between currency and reserves ($\eta = 1$), as seen in a traditional central banking system, is always optimal. This is because the traditional one-to-one exchange rate enables the central bank to eliminate costly theft without reducing welfare. Moreover, optimality is achieved when the central bank sets the nominal interest rate on reserves R^m at the base lower bound (the ELB level under $\eta = 1$). Given that the base lower bound on R^m is above zero, the consumption quantity in deposit-based DM transactions exceeds those in currency-based transactions for any R^m above the base lower bound. Consequently, reducing R^m until it reaches the lower bound maximizes welfare as it allows consumption smoothing across DM transactions.

A.3.6 Discussion

When considering real-world payment systems, the baseline model and the extended model capture different aspects. The baseline model effectively reflects the distinction between cash transactions and electronic transactions, where some transactions can only be made with currency due to various reasons such as accessibility or privacy concerns, while others solely involve deposit claims, as seen in online transactions. In contrast, the extended model addresses the substitutability between currency and deposit claims but does not consider electronic transactions where only deposit claims are accepted. Therefore, neither model fully captures the complexity and diversity of real-world payment systems, but both provide valuable insights into the implications of implementing a non-par exchange rate policy for the ELB and welfare.

A.4 Quantitative Analysis

Theoretically, introducing a non-par exchange rate between currency and reserves can lead to welfare losses by incentivizing costly theft and distorting the equilibrium allocation. To quantify these welfare losses, I calibrate the baseline model to the U.S. economy and conduct a counterfactual analysis to evaluate the effects of introducing a non-par exchange rate between currency and reserves.

A.4.1 Calibration

I consider an annual model where the utility function in the DM takes the form $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$. When calibrating the baseline model to data, I exclude the cost of theft ε because a one-to-one exchange rate between currency and reserves implies no theft occurring in the model.[36] Then, there are eight parameters to calibrate: σ (the curvature of DM consumption), β (the discount factor), ρ (the fraction of currency transactions in the DM), μ (the fixed cost

[36] To quantify the welfare cost arising from an increase in theft, the cost of theft parameter ε needs to be calibrated outside the model. However, to the best of my knowledge, there is no available data that allows for the direct measurement of this parameter.

Parameters	Values	Calibration targets	Sources
β	0.96	Standard in literature	
R^m	1.0025	Avg. interest rate on reserves: 0.25%	FRED
γ	0.00	Lowest target range for fed funds rate: 0-0.25%	FRED
σ	0.17	Money demand elasticity (1959-2007): -4.19	FRED
ρ	0.17	Currency to M1 ratio: 17.22%	FRED; Lucas and Nicolini (2015)
v	1.13	Avg. locally-held public debt to GDP: 66.73%	FRED
δ	0.45	Avg. inflation rate: 1.06%	FRED
μ	0.01	Fixed storage cost: 2% of currency payments	Author's assumption

Table 3: Calibration results

of storing currency), γ (the proportional cost of storing currency), δ (the fraction of assets private banks can abscond with), R^m (the nominal interest rate on reserves), and v (the value of the consolidated government's liabilities held by the public).

Table 3 summarizes the calibration results along with the target moments, which are mostly constructed from U.S. data for 2013-2015. This period is chosen because, for the purpose of this exercise, it is suitable to consider a timeframe when the policy rate was close to zero. Also, key variables such as the nominal interest rate on reserves and domestically-held public debt to GDP were stable during this period.

There are three parameters calibrated externally. The discount factor β is given by $\beta = 0.96$. From Federal Reserve Economic Data (FRED), the nominal interest rate on reserves was 0.25 percent over the period 2013-2015 ($R^m = 1.0025$). Finally, the lowest target range for the federal funds rate has been between 0 and 0.25 percent since 1954. Although this does not imply that the proportional cost of storing currency is zero, the proportional cost γ is assumed to be zero for convenience.[37]

Calibrating σ (the curvature of DM consumption) involves matching the elasticity of money demand in the model with the empirical money demand elasticity obtained from the data. Estimating the money demand elasticity requires a longer time series of data,

[37] The proportional cost of storing currency implies that the ELB on the nominal interest rate can be negative. However, the Federal Reserve might have faced legal and political issues with implementing negative nominal interest rates. As negative rates have not been explored in the U.S., it seems difficult to calibrate the proportional cost of storing currency with this model.

so I select the time period from 1959 to 2007.[38] Using data on currency in circulation and nominal GDP (from FRED), I calculate the currency-to-GDP ratios. Then, the money demand elasticity can be estimated using Moody's AAA corporate bond yields (from FRED) and the currency-to-GDP ratios, resulting in an estimated elasticity of -4.19.[39]

Then, I jointly calibrate four parameters: the curvature of DM consumption σ, the fraction of currency transactions in the DM ρ, the value of government liabilities held by the public v, the fraction of bank assets that can be absconded δ, and the fixed cost of storing currency μ. The curvature parameter σ is calibrated to match the estimated money demand elasticity. Using currency-in-circulation data from FRED and the new M1 series from Lucas and Nicolini (2015), I calibrate the fraction of currency transactions in the DM ρ until the model generates the observed currency-to-M1 ratio. I use domestically-held public debt to GDP from FRED to calibrate the value of publicly-held government liabilities v.[40] Another variable I use to calibrate parameters is the inflation rate. Along with other parameters, the fraction of assets that can be absconded δ is calibrated so that the model generates an inflation rate consistent with the observed rate of 1.06 percent. Finally, I set the fixed cost of storing currency μ to be 2 percent of cash payments.[41]

A.4.2 Counterfactual Analysis

I consider three different scenarios where the fixed cost of theft ε is set to (i) 2.5 percent, (ii) 5 percent, and (iii) 10 percent of the current consumption level. With the calibrated parameters

[38] In the aftermath of the global financial crisis in 2007-2008, the demand for currency increased possibly due to non-transactional purposes. To calculate the elasticity of money demand specifically for transactions, I exclude post-crisis data, following the approach of Chiu, Davoodalhosseini, Jiang, and Zhu (2022) and Altermatt (2022).

[39] The interest rate on liquid bonds (e.g., the 3-month Treasury Bill rate) may fluctuate due to changes in the liquidity premium. To exclude such a possibility, I consider the AAA corporate bond yield as the nominal interest rate on illiquid bonds and $\frac{\pi}{\beta} - 1$ as its theoretical counterpart.

[40] I define domestically-held public debt as the total public debt net of public debt held by foreign and international investors.

[41] Under this assumption, each buyer pays approximately 2 percent more to purchase goods in currency transactions, compensating for the seller's storage cost. While the fixed storage cost may indeed deviate from 2 percent, adjusting it within the range of 0 percent to 10 percent does not significantly affect the outcomes of the counterfactual analysis.

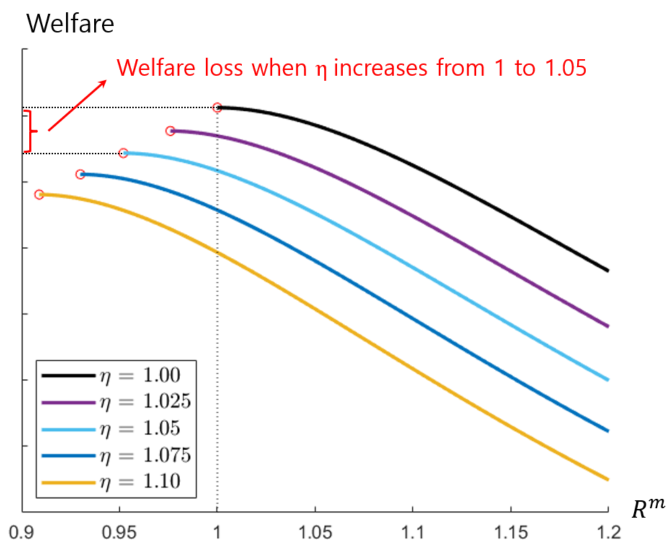

Figure 6: Monetary policy (R^m, η) and welfare

and each value of ε, I vary the non-par exchange rate η and compute the corresponding nominal interest rate on reserves that maximizes welfare, denoted by R^*_η. As depicted in Figure 6, an increase in η reduces the ELB on the nominal interest rate. However, the welfare level under the optimal nominal interest rate R^*_η decreases as η rises.

My approach to quantifying the welfare cost of increasing η involves measuring how much consumption individuals would need to be compensated to tolerate the non-par exchange rate η. Suppose that, for any given η, the central bank sets the nominal interest rate at the optimal level, i.e., $R^m = R^*_\eta$. Then, the welfare measure can be expressed as

$$\mathcal{W}(R^m = R^*_\eta, \eta) = \rho[u(x^c) - x^c] + (1-\rho)[u(x^d) - x^d] - \alpha^b \varepsilon.$$

If the consumption quantities in the DM are adjusted by a factor Δ, the adjusted level of welfare can be expressed as

$$\mathcal{W}_\Delta(R^m = R^*_\eta, \eta) = \rho[u(\Delta x^c) - x^c] + (1-\rho)[u(\Delta x^d) - x^d] - \alpha^b \varepsilon.$$

Then, I compute the value Δ_η that solves $\mathcal{W}_{\Delta_\eta}(R^m = R^*_\eta, \eta > 1) = \mathcal{W}(R^m = R^*_\eta, \eta = 1)$.

η	ELB	R^*_η	$(\Delta_\eta - 1) \times 100$		
			$\varepsilon = 2.5\%$	$\varepsilon = 5\%$	$\varepsilon = 10\%$
1.00	1.000	1.000	-	-	-
1.025	0.976	0.976	0.0561	0.1118	0.2236
1.05	0.952	0.952	0.1095	0.2183	0.4367
1.075	0.930	0.930	0.1604	0.3199	0.6399
1.10	0.909	0.909	0.2091	0.4168	0.8339

Table 4: ELB, optimal interest rate, and the welfare cost of reducing the ELB

The welfare cost of introducing η can be measured as $\Delta_\eta - 1$ percent of consumption. If individuals are compensated with this amount of consumption, they would be indifferent between the two policy choices: a one-to-one exchange rate and a non-par exchange rate.

Table 4 presents the ELB, the optimal nominal interest rate R^*_η, and the welfare cost of introducing a non-par exchange rate η for various fixed costs of theft ε. Notably, an increase in η reduces both the ELB and the optimal interest rate, regardless of the cost of theft. Recall that, with a fixed cost of storing currency close to zero ($\mu \approx 0$), the optimal monetary policy can be characterized by a modified Friedman rule ($\eta R^m \approx 1$). Thus, an increase in η leads to a decrease in the optimal nominal interest rate R^*_η. Assuming that monetary policy is conducted optimally for any given non-par exchange rate η, there would be no distortion in the equilibrium prices and allocations.

However, introducing a non-par exchange rate η increases the aggregate cost of theft in equilibrium. For instance, if the fixed cost of theft ε is 2.5 percent of the current consumption level, increasing η by 5 percent and 10 percent would cost, respectively, 0.11 percent and 0.21 percent of consumption. If ε is 5 percent of the current consumption level, the corresponding increases in η would cost 0.22 percent and 0.42 percent of consumption. Finally, with ε set at 10 percent of the current consumption level, the increases in η by 5 percent and 10 percent would cost 0.44 percent and 0.84 percent of consumption, respectively.[42]

The welfare cost of introducing a non-par exchange rate can be better understood by

[42]Using different values for the fixed cost of storing currency μ does not significantly change the result. For example, if μ is 10 percent of cash payments and ε is 5 percent of the current consumption level, increasing η by 5 percent and 10 percent would cost, respectively, 0.2185 percent and 0.4172 percent of consumption.

comparing it with estimates for the welfare cost of another policy frequently discussed in the literature: the welfare cost of 10 percent inflation. Estimates for the welfare cost of 10 percent inflation are typically around 1 percent of consumption. Therefore, the welfare cost of introducing a non-par exchange rate seems significant.[43] It is important to note that the welfare cost of using a non-par exchange rate critically depends on the fixed cost of investing in the theft technology ε. As ε increases, the welfare cost also increases proportionally.

A.5 Omitted Proofs

Proof of Lemma 1: First, consider the case where $\eta = 1$. If theft does not occur in equilibrium ($\alpha^b = 0$), sellers would be indifferent between depositing their currency with the central bank and engaging in side-trading with private banks ($\alpha^s \in [0, 1]$). For this equilibrium to be incentive-compatible, the cost of theft ε must be sufficiently high to satisfy $\varepsilon \geq \frac{\rho\alpha^s\eta c'}{\pi}$. Moreover, if $\varepsilon > \frac{\rho\eta c'}{\pi}$, investing in theft ($\alpha^b = 0$) is never optimal for buyers for any $\alpha^s \in [0, 1]$. If $\varepsilon = \frac{\rho\eta c'}{\pi}$, buyers become indifferent between investing in theft and not investing, although only $\alpha^b = 0$ can be sustainable in equilibrium. Conversely, if $\varepsilon < \frac{\rho\eta c'}{\pi}$, the fraction of sellers carrying currency into the TS must be sufficiently low to prevent theft. For ε to be sufficiently high to satisfy $\varepsilon \geq \frac{\rho\alpha^s\eta c'}{\pi}$, the fraction α^s must fall within the range of $[0, \bar{\alpha}^s]$, where $\bar{\alpha}^s = \frac{\varepsilon\pi}{\rho\eta c'}$. Therefore, when $\eta = 1$, there exist a continuum of no-theft equilibria with $\alpha^b = 0$ and $\alpha^s \in [0, 1]$ for $\varepsilon \geq \rho\eta c^s$, and with $\alpha^b = 0$ and $\alpha^s \in [0, \bar{\alpha}^s]$ for $\varepsilon < \frac{\rho\eta c'}{\pi}$, where $\bar{\alpha}^s = \frac{\varepsilon\pi}{\rho\eta c'}$. However, a theft equilibrium does not exist in this case. This is because, when buyers steal currency ($\alpha^b \in (0, 1]$), no sellers would carry their currency into the TS ($\alpha^s = 0$), which disincentivizes buyers from investing in the costly theft technology.

Next, I consider the case where $\eta > 1$. If no theft occurs in equilibrium ($\alpha^b = 0$), sellers would prefer side-trading their currency with private banks over depositing it with the central bank ($\alpha^s = 1$). A necessary condition for this equilibrium to exist is $\varepsilon \geq \frac{\rho\eta c'}{\pi}$. Therefore,

[43] For instance, the welfare cost of increasing inflation from 0 percent to 10 percent is reported as 0.62 percent of consumption in Chiu and Molico (2010), 0.87 percent in Lucas (2000), and 1.32 percent (assuming that buyers make a take-it-or-leave-it offer to sellers) in Lagos and Wright (2005), among other studies.

a unique equilibrium exists with $\alpha^b = 0$ and $\alpha^s = 1$ for $\varepsilon \geq \frac{\rho\eta c'}{\pi}$. If $\varepsilon < \frac{\rho\eta c'}{\pi}$, buyers would choose to incur costs to acquire theft technology ($\alpha^b = 1$). However, cases with $\alpha^b = 1$ (where buyers strictly prefer to invest in theft) cannot be sustained in equilibrium because $\alpha^b = 1$ would lead to $\alpha^s = 0$, discouraging costly theft. Thus, in equilibrium, both buyers and sellers must be indifferent between their options. From (7) and (8), I obtain:

$$\alpha^b = \frac{\eta - 1}{\eta}, \tag{65}$$

$$\alpha^s = \frac{\varepsilon\pi}{\rho\eta c'}. \tag{66}$$

Hence, for $\varepsilon < \frac{\rho\eta c'}{\pi}$, there exists a unique equilibrium that satisfies the above equations.

Proof of Proposition 1: In equilibrium, equations (27) and (29) solve for x^c and x^d. For the comparative statics analysis with respect to R^m, I totally differentiate (27) and (29) and evaluate the derivatives of x^c and x^d for $(R^m, \eta) = (1, 1)$ and $\mu = 0$, which yield:

$$\frac{dx^c}{dR^m} = \frac{(1-\rho)[(1-\delta)u'(x)+\delta][(1-\delta)(1-\sigma)u'(x)+\delta]}{u''(x)[(1-\delta)(1-\sigma)u'(x)+\delta+\rho\mu(1-\delta)u''(x)]} < 0,$$
$$\frac{dx^d}{dR^m} = \frac{-\rho[(1-\delta)u'(x)+\delta][(1-\delta)(1-\sigma)u'(x)+\delta+\mu(1-\delta)u''(x)]}{u''(x)[(1-\delta)(1-\sigma)u'(x)+\delta+\rho\mu(1-\delta)u''(x)]} > 0,$$

where $x^c = x^d = x$ and $\sigma \equiv -\frac{xu''(x)}{u'(x)} \in (0, 1)$. It follows immediately that from (26) and (28), r^m, r^b, and π increase, while from the first argument in (31), the ELB remains unchanged.

Now, I turn attention to the comparative statics analysis with respect to η. Then, evaluating the derivatives of x^c and x^d with respect to η for $(R^m, \eta) = (1, 1)$ and $\mu = 0$ yields:

$$\frac{dx^c}{d\eta} = -\frac{\delta\{\rho\sigma u'(x)+(1-\rho)[(1-\delta)(1-\sigma)u'(x)+\delta][u'(x)-1]-\mu\rho u''(x)\}}{u''(x)[(1-\delta)(1-\sigma)u'(x)+\delta+\rho\mu(1-\delta)u''(x)]} > 0,$$
$$\frac{dx^d}{d\eta} = \frac{\rho\delta[(1-\sigma)u'(x)-1+\mu u''(x)][(1-\delta)u'(x)+\delta]}{u''(x)[(1-\delta)(1-\sigma)u'(x)+\delta+\rho\mu(1-\delta)u''(x)]}.$$

Additionally, evaluating equation (29) for $(R^m, \eta) = (1, 1)$ and $\mu = 0$ yields:

$$\left[u'(x) + \frac{\delta}{1-\delta}\right](x + \rho\mu) = v, \tag{67}$$

where x is increasing in v. The collateral constraint (67) does not bind in equilibrium if

$$v \geq \frac{x^* + \rho\mu}{1-\delta}, \tag{68}$$

where x^* is the first-best quantity of consumption in the DM that satisfies $u'(x^*) = 1$. Let $\bar{v}$ denote the right-hand side of inequality (68), $\hat{x}$ denote the solution to $u'(x) = \frac{1}{1-\sigma}$, and $\hat{v}$ denote the solution to (67) when $x = \hat{x}$. Then, the derivatives of x^d with respect to η can be written as

$$\begin{aligned} \frac{dx^d}{d\eta} &\leq 0, \qquad \text{if} \quad v \in (0, \hat{v}] \\ \frac{dx^d}{d\eta} &> 0. \qquad \text{if} \quad v \in (\hat{v}, \bar{v}) \end{aligned}$$

Thus, from (26), an increase in η decreases r^m and r^b for $v \in (0, \hat{v}]$ and increases r^m and r^b for $v \in (\hat{v}, \bar{v})$. From (28) or

$$\pi = \beta R^m \left[u'(x^d) - \delta u'(x^d) + \delta\right],$$

an increase in η increases π for $v \in (0, \hat{v}]$ and decreases π for $v \in (\hat{v}, \bar{v})$. Finally, from the first argument in (31), the ELB falls.

Proof of Proposition 2: The DM consumption quantities, x^c and x^d, are determined by equations (33) and (35). Equation (35) can be expressed as

$$F(x^c, x^d) = v, \tag{69}$$

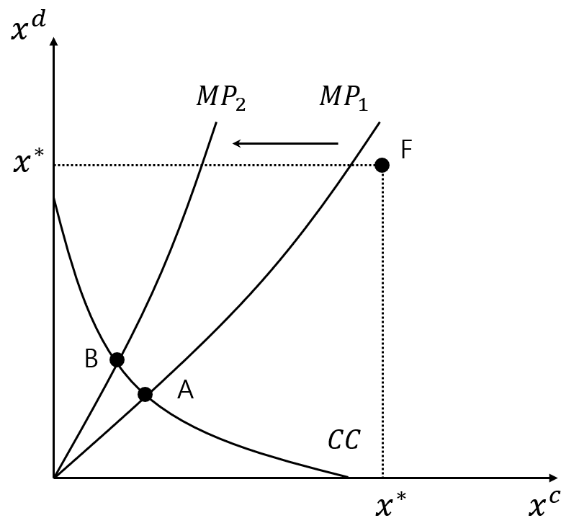

Figure 7: Effects of monetary policy (R^m, η)

and I can show that the function $F(\cdot,\cdot)$ is strictly increasing in both $0 \leq x^c < x^*$ and $0 \leq x^d < x^*$, as $-x\frac{u''(x)}{u'(x)} < 1$. Again, x^* denotes the first-best quantity of consumption in the DM that satisfies $u'(x^*) = 1$. Based on this property, equation (69) can be depicted by a downward-sloping locus in the (x^c, x^d) space, given v. Conversely, equation (33) can be depicted by an upward-sloping locus in the (x^c, x^d) space, given (R^m, η).

For the comparative statics analysis, suppose the central bank raises R^m while holding η constant. As illustrated in Figure 7, the MP curve, representing equation (33), shifts to the left while the CC curve, representing equation (69), remains unaffected. Thus, x^c decreases, and x^d increases. Then, from (32) and (34), π rises, and real interest rates (r^m, r^b) rise. From (36) and (38), α^s increases, while α^b and the ELB remain unchanged.

Next, suppose that the central bank raises η while holding R^m constant. This monetary intervention also shifts the MP curve leftward, while leaving the CC curve unaffected. Consequently, x^c decreases, and x^d increases. Then, from (34), real interest rates (r^m, r^b) rise. Using (33), equation (32) can be written as:

$$\pi = \beta R^m \left[u'(x^d) - \delta u'(x^d) + \delta\right],$$

so π falls. From (36) and (38), α^b increases, and the ELB decreases. However, the effect on α^s is ambiguous since ηx^c can increase or decrease depending on parameters.

Proof of Proposition 3: The proof consists of two steps. First, I will identify monetary policies that maximize the welfare measure $\mathcal{W}$, assuming α^b is exogenously given. Then, I will determine the optimal monetary policy considering α^b as endogenously determined in response to monetary policy.

In the first step, I solve the following maximization problem given $\alpha^b \in [0, 1]$:

$$\max_{(R^m,\eta)} \rho\left[u\left(x^c\right) - x^c\right] + (1-\rho)\left[u\left(x^d\right) - x^d\right] - \alpha^b \varepsilon \tag{70}$$

subject to

$$\eta R^m = \frac{u'\left(x^c\right) - \delta u'\left(x^d\right) + \delta}{u'\left(x^d\right) - \delta u'\left(x^d\right) + \delta}, \tag{71}$$

$$\left[u'(x^c) + \frac{\delta}{1-\delta}\right]\rho(x^c + \mu) + \left[u'\left(x^d\right) + \frac{\delta}{1-\delta}\right](1-\rho)x^d = v, \tag{72}$$

$$R^m \geq \frac{1}{\eta + \beta\gamma}, \quad \eta \geq 1 \tag{73}$$

Here, a monetary policy measure relevant to welfare in equilibrium is ηR^m, denoted as $\Omega \equiv \eta R^m$. Hence, from (73), Ω must satisfy $\Omega \geq \frac{\eta}{\eta+\beta\gamma}$. Differentiating the objective (70) with respect to Ω yields:

$$\frac{d\mathcal{W}}{d\Omega} = \rho\left[u'(x^c) - 1\right]\frac{dx^c}{d\Omega} + (1-\rho)\left[u'(x^d) - 1\right]\frac{dx^d}{d\Omega}. \tag{74}$$

Let $\sigma = -\frac{xu''(x)}{u'(x)}$. Then, from total differentiation of (71) and (72), I derive:

$$\frac{dx^c}{d\Omega} = \frac{(1-\rho)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Phi} < 0, \tag{75}$$

$$\frac{dx^d}{d\Omega} = \frac{-\rho\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta} + \mu u''(x^c)\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Phi} > 0, \tag{76}$$

where

$$\Phi = (1-\rho)u''(x^c)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] \\ + \rho u''(x^d)\left[(1-\delta)\Omega + \delta\right]\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta} + \mu u''(x^c)\right] < 0,$$

for sufficiently low μ. A monetary policy Ω attains a local optimum if the resulting consumption quantities x^c and x^d lead to $\frac{d\mathcal{W}}{d\Omega} = 0$. From (74)-(76), I can characterize the optimal allocation x^c and x^d as follows:

$$\frac{d\mathcal{W}}{d\Omega} = 0,$$

$$\Leftrightarrow \left[u'(x^c) - 1\right]\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] - \left[u'(x^d) - 1\right]\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta} + \mu u''(x^c)\right] = 0,$$

$$\Rightarrow \left[u'(x^c) - 1\right]\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] \leq \left[u'(x^d) - 1\right]\left[(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta}\right],$$

$$\Leftrightarrow \frac{u'(x^c) - 1}{(1-\sigma)u'(x^c) + \frac{\delta}{1-\delta}} \leq \frac{u'(x^d) - 1}{(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}}.$$

Since the function $F(x) = \frac{u'(x)-1}{(1-\sigma)u'(x)+\frac{\delta}{1-\delta}}$ is strictly decreasing in x, the above inequality is equivalent to $x^c \geq x^d$. Note that from (71) $x^c = x^d$ if $\Omega = 1$, and that as Ω rises, x^c decreases and x^d increases. Therefore, to achieve $x^c \geq x^d$, the optimal monetary policy Ω must satisfy $\Omega \leq 1$, which holds with equality if and only if $\mu = 0$.

In the first step, I have shown that an optimal monetary policy must be a combination of (R^m, η) such that $\eta R^m \leq 1$. All optimal combinations of (R^m, η) lead to the same gains from trade in the DM, $\rho\left[u\left(x^c\right) - x^c\right] + (1-\rho)\left[u\left(x^d\right) - x^d\right]$. However, as the non-par exchange rate η rises, the fraction of buyers who choose to steal currency in the TS α^b increases (36), leading to a increase in the total cost of theft in the TS. Therefore, the optimal monetary policy is given by $\eta = 1$ and $R^m \leq 1$.

Even if the optimal R^m is constrained by (73), the optimal interest rate is the lower bound $\frac{1}{\eta+\beta\gamma}$. An increase in η allows R^m to decrease further, but it acts to increase the current level of Ω. So, an increase in η cannot increase the sum of surpluses from trade in the CM and the DM, represented by the first two terms in the welfare measure (70), while increasing the total theft cost in the TS.

Proof of Proposition 4: Suppose the cost of theft ε is sufficiently high to prevent theft in equilibrium ($\alpha^b = 0$). To show that social welfare increases with η, consider the scenario where the central bank sets the nominal interest rate R^m to achieve $x^c = x^d = x$ given a non-par exchange rate η. While this policy may not be optimal, it helps us understand the optimal level of η. When $x^c = x^d = x$, equations (27) and (29) can be written as

$$R^m = R^b = \frac{\eta u'(x) - \delta u'(x) + \delta}{\eta \left[u'(x) - \delta u'(x) + \delta\right]}, \tag{77}$$

$$u'(x)\left[x + \rho\mu\right] + \frac{\left[\rho + (1-\rho)\eta\right]\delta x + \rho\delta\mu}{(1-\delta)\eta} = v. \tag{78}$$

In this case, equation (78) solves for x, and then (77) solves for R^m given η. Furthermore, if the value of the consolidated government debt v is sufficiently low, or

$$v \leq \frac{\left[(1-\delta\rho)\eta + \delta\rho\right]x^* + \rho\mu\left[(1-\delta)\eta + \delta\right]}{(1-\delta)\eta},$$

then x increases with η for a sufficiently low μ.[44] Since the welfare measure can be written as $\mathcal{W} = u(x) - x$, an increase in η effectively increases welfare as long as the nominal interest rate R^m can be adjusted to achieve $x^c = x^d = x$.

However, according to Proposition 1, an increase in η must be accompanied by an increase in R^m to maintain the same consumption quantities in the two types of DM meetings, implying that the choice of R^m is not constrained by the ELB. Although the optimal R^m may not satisfy $x^c = x^d$, the fact that social welfare increases with η remains unchanged. Finally, η must be sufficiently low to prevent buyers from investing in theft technology. Therefore, at the optimum, η is chosen such that buyers are indifferent between stealing

[44] Here, x^* is the solution to $u'(x) = 1$.

currency and not stealing.

Suppose further that there is no fixed cost of holding currency at the beginning of the TS, i.e., $\mu = 0$. Consider the following maximization problem given $\eta \geq 1$:

$$\max_{(R^m, \eta)} \rho \left[u\left(x^c\right) - x^c\right] + (1-\rho)\left[u\left(x^d\right) - x^d\right] \tag{79}$$

subject to

$$\eta R^m = \frac{\eta u'\left(x^c\right) - \delta u'\left(x^d\right) + \delta}{u'\left(x^d\right) - \delta u'\left(x^d\right) + \delta}, \tag{80}$$

$$\left[u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\rho x^c + \left[u'\left(x^d\right) + \frac{\delta}{1-\delta}\right](1-\rho)x^d = v, \tag{81}$$

$$R^m \geq \max\left\{\frac{1}{\eta + \beta\gamma}, \frac{\eta}{(\eta+\beta\gamma)[(1-\delta)u'(x^d) + \delta]}\right\}. \tag{82}$$

Differentiating the objective (79) with respect to R^m yields:

$$\frac{d\mathcal{W}}{dR^m} = \rho\left[u'(x^c) - 1\right]\frac{dx^c}{dR^m} + (1-\rho)\left[u'(x^d) - 1\right]\frac{dx^d}{dR^m}. \tag{83}$$

Letting $\sigma = -\frac{xu''(x)}{u'(x)}$ and totally differentiating (80) and (81) gives:

$$\frac{dx^c}{dR^m} = \frac{\eta(1-\rho)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Lambda} < 0,$$

$$\frac{dx^d}{dR^m} = \frac{-\eta\rho\left[(1-\sigma)u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\left[(1-\delta)u'(x^d) + \delta\right]}{\Lambda} > 0,$$

where

$$\Lambda = (1-\rho)\eta u''(x^c)\left[(1-\sigma)u'(x^d) + \frac{\delta}{1-\delta}\right] + \rho u''(x^d)\left[(1-\sigma)u'(x^c) + \frac{\delta}{(1-\delta)\eta}\right]\left[\eta R^m(1-\delta) + \delta\right] < 0.$$

Then, I can evaluate the derivative of $\mathcal{W}$, or equation (83), for $\eta R^m = 1$. Noting that $\eta u'(x^c) = u'(x^d)$ from (80), I obtain

$$\left.\frac{d\mathcal{W}}{dR^m}\right|_{\eta R^m=1} = \frac{\eta(1-\eta)\rho(1-\rho)\left[(1-\delta)u'(x^d)+\delta\right]}{\eta^2(1-\rho)u''(x^c)+\rho u''(x^d)} \geq 0, \tag{84}$$

implying that $\eta R^m \geq 1$ at an optimum.

Next, differentiating the objective (79) with respect to η gives:

$$\frac{d\mathcal{W}}{d\eta} = \rho\left[u'(x^c)-1\right]\frac{dx^c}{d\eta} + (1-\rho)\left[u'(x^d)-1\right]\frac{dx^d}{d\eta}. \tag{85}$$

I totally differentiate (80) and (81) to derive $\frac{dx^c}{d\eta}$ and $\frac{dx^d}{d\eta}$, and then evaluate the derivatives for $\eta R^m = 1$, which yields:

$$\begin{aligned}
\frac{dx^c}{d\eta} &= \frac{\delta\rho x^c u''(x^d) - \delta\eta(1-\rho)\left[u'(x^d)-1\right]\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]}{\eta\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]\left[\rho u''(x^d)+\eta^2(1-\rho)u''(x^c)\right]} > 0,\\
\frac{dx^d}{d\eta} &= \frac{\delta\rho\left\{\eta x^c u''(x^c) + \left[u'(x^d)-1\right]\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]\right\}}{\eta\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]\left[\rho u''(x^d)+\eta^2(1-\rho)u''(x^c)\right]}.
\end{aligned}$$

Substituting the above expressions into (85) gives:

$$\left.\frac{d\mathcal{W}}{d\eta}\right|_{\eta R^m=1} = \frac{\Gamma + \rho\delta x^c\left\{\rho u''(x^d)\left[u'(x^c)-1\right] + \eta(1-\rho)u''(x^c)\left[u'(x^d)-1\right]\right\}}{\eta\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right]\left[\rho u''(x^d)+\eta^2(1-\rho)u''(x^c)\right]}, \tag{86}$$

where

$$\Gamma = \rho\delta(1-\rho)(\eta-1)\left[u'(x^d)-1\right]\left[(1-\delta)(1-\sigma)u'(x^d)+\delta\right] \geq 0.$$

From (86), the derivative of $\mathcal{W}$ is strictly positive if $\eta = R^m = 1$, i.e., $\left.\frac{d\mathcal{W}}{d\eta}\right|_{\eta=R^m=1} > 0$. This implies that the monetary policy at $\eta = R^m = 1$ is not optimal. Therefore, from (84) and (86), I conclude that the optimal monetary policy is characterized by $\eta R^m > 1$.

Proof of Proposition A.3.1: Suppose that $\theta = 0$ in equilibrium. From equations (40)-(42) and (47),

$$\eta R^m \left[u'(x^d) - \delta u'(x^d) + \delta\right] = u'(x^o), \tag{87}$$

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \tag{88}$$

where (x^c, x^d) are the off-equilibrium consumption quantities in the DM, if a buyer were to participate in banking contracts. It can be shown that $\left|\frac{d[u'(x^d)-\delta u'(x^d)+\delta]}{d[\eta R^m]}\right| < 1$, so from (87) x^o increases with a decrease in ηR^m. However, the limited quantity of collateral (represented by $v < x^* + \mu$) implies that, from (57), the highest possible quantity for x^o is $\bar{x}$ that solves $(\bar{x} + \mu)u'(\bar{x}) = v$, and $\bar{x} < x^*$. So, any ηR^m that leads to x^o higher than $\bar{x}$ cannot be supported in equilibrium, implying that, from (87),

$$R^m \geq \frac{u'(\bar{x})}{\eta\left[u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta\right]}, \tag{89}$$

where $\underline{x}^d$ is the off-equilibrium consumption quantity in deposit-based DM transactions that is consistent with (88). Also, any R^m higher than the lower bound in (89) results in x^o such that $(x^o + \beta\mu)u'(x^o) < v$. This implies that $0 < \theta \leq 1$ and $U^b \geq U^o$ in equilibrium. So, by continuity, the off-equilibrium consumption quantities (x^c, x^d) when $R^m = \frac{u'(\bar{x})}{\eta\left[u'(\underline{x}^d)-\delta u'(\underline{x}^d)+\delta\right]}$ must satisfy (88) and $U^b = U^o$ from (49)-(50) given $x^o = \bar{x}$.

Recall that the nominal interest rate R^m must also satisfy (38), the no-arbitrage condition for agents from holding currency across periods. To prove that the lower bound on R^m given in (89) is always higher than the lower bound in (38), I claim that the following condition must hold in equilibrium if the fixed cost of holding currency μ is close to zero:

$$\frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta} \geq 1. \tag{90}$$

To prove this claim, suppose $\eta R^m = 1$. In this case, deposit contracts allow buyers to consume the same quantity of goods across two types of DM transactions, i.e., $x^c = x^d = x$. Then, the quantity of DM consumption for buyers opting out of deposit contracts is higher than the quantity for those holding deposit contracts ($x^o > x$) because, from (87)-(88),

$$u'(x^o) = (1-\delta)u'(x) + \delta.$$

This implies that the expected utility for buyers opting out of contracts is higher than the expected utility for those opting in, because from (49)-(50),

$$U^o - U^b = [u(x^o) - x^o u'(x^o)] - [u(x) - xu'(x)] - \mu\,[u'(x^o) - \rho u'(x)] > 0,$$

for a sufficiently low μ. Thus, the policy combination of $\eta R^m = 1$ leads to complete disintermediation ($\theta = 0$), which implies that the lower bound, $\frac{u'(\bar{x})}{u'(\underline{x}^d)-\delta u'(\underline{x}^d)+\delta}$, must be higher than or equal to one. Also, $\frac{u'(\bar{x})}{\eta\left[u'(\underline{x}^d)-\delta u'(\underline{x}^d)+\delta\right]} > \frac{1}{\eta+\beta\gamma}$ for any $\eta \geq 1$ because, as η rises, $\eta + \beta\gamma$ increases more than $\eta\left[u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta\right]$. Therefore, the ELB on the nominal interest rate is always determined by (89).

Proof of Proposition A.3.2: First, note that an equilibrium with $\theta = 0$ exists only when R^m is set at the ELB defined in (55). As mentioned in the Proof of Proposition A.3.1, any R^m higher than the ELB implies $x^o < \bar{x}$, where $\bar{x}u'(\bar{x}) = v$. Since $x^o u'(x^o) < v$ and the collateral constraint must bind in an equilibrium where v is sufficiently low, there must exist some buyers participating in banking contracts, i.e., $\theta > 0$. Thus, for any R^m higher than the ELB, $\theta > 0$ in equilibrium.

Next, consider an equilibrium with $0<\theta<1$. In this case, (x^c, x^d, x^o, θ) must satisfy:

$$\eta R^m = \frac{u'(x^c) - \delta u'(x^d) + \delta}{u'(x^d) - \delta u'(x^d) + \delta}, \tag{91}$$

$$(1-\rho)\theta x^d \left[u'(x^d) + \frac{\delta}{1-\delta}\right] + \rho\theta x^c \left[u'(x^c) + \frac{\delta}{1-\delta}\right] + (1-\theta)x^o u'(x^o) = v, \tag{92}$$

$$u'(x^o) = u'(x^c) - \delta u'(x^d) + \delta, \tag{93}$$

$$\rho\left[u(x^c) - x^c u'(x^c)\right] + (1-\rho)\left[u(x^d) - x^d u'(x^d)\right] = u(x^o) - x^o u'(x^o). \tag{94}$$

Suppose that there is an increase in ηR^m. Then, from (91), x^c decreases while x^d increases. From (93), x^o decreases with the decrease in x^c and the increase in x^d. Also, from (55) and (93), a necessary condition for this equilibrium to exist is $x^c < x^d$, implying that U^b decreases as x^c falls and x^d rises. As the left-hand side of (94) (representing U^b) decreases, x^o must fall in equilibrium, consistent with (93). Then, from (92), θ must rise in equilibrium. The effects of an increase in ηR^m on $(\pi, \alpha^b, \alpha^s)$ can be derived from (58), (61), and (62).

Given that an increase in ηR^m leads to a decrease in x^c and x^o and an increase in x^d and θ, there exists a ηR^m, denoted by $\bar{\Omega}$, that satisfies equation (91), with x^c and x^d that consist of the solution to equations (92)-(94) when $\theta = 1$. Therefore, I can conclude that, in equilibrium, $0 \leq \theta < 1$ if $\frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta} \leq \eta R^m < \bar{\Omega}$ and $\theta = 1$ if $\eta R^m \geq \bar{\Omega}$, where $\bar{x}$ and $\underline{x}^d$ are the quantities defined in the Proof of Proposition A.3.1.

Proof of Proposition A.3.3: In the Proof of Proposition A.3.2, I have shown that, for any $\eta R^m > \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$, the fraction θ is positive and $x^c < x^d$ in equilibrium. This implies that both the left-hand and the right-hand side of (94) increase (i.e., both U^b and U^o increase) as x^c and x^o rise and x^d falls. So, given η, a decrease in R^m increases the welfare measure $\mathcal{W}$ by increasing x^c and x^o and decreasing x^d and θ. Then, by continuity, the maximum $\mathcal{W}$ can be obtained when $\eta R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$ given η. However, if the central bank conducts monetary policy (R^m, η) such that $\eta R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$, a higher η only implies a higher α^b without increasing the sum of surpluses from trade in the CM and the DM. As a higher

α^b leads to a larger total cost of theft, the welfare measure $\mathcal{W}$ is maximized if and only if $\eta = 1$ and $R^m = \frac{u'(\bar{x})}{u'(\underline{x}^d) - \delta u'(\underline{x}^d) + \delta}$.